

# Strong convergence rate in averaging principle for stochastic hyperbolic-parabolic equations with two time-scales

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## Abstract

In this article, we investigate averaging principle for stochastic hyperbolic-parabolic equations with slow and fast time-scales, in which both the slow and fast components are perturbed by multiplicative noise. Particularly, we prove that the rate of strong convergence for the slow component to the averaged dynamics is of order  $1/2$ . This extends the results for finite dimensional stochastic dynamical systems to the infinite dimension case.

**Keywords:** Stochastic hyperbolic-parabolic equations; averaging principle; invariant measure and ergodicity; strong convergence rate.

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# 1 Introduction

This paper, which is a sequel to [12], is devoted to the strong convergence rate for averaging principle for a hyperbolic-parabolic coupled systems of partial differential equations with two widely separated timescales.

Let  $D = (0, L) \subset \mathbb{R}$  be a bounded open interval. For fixed  $T_0 > 0$ , we are concerned with the following stochastic hyperbolic-parabolic equation with multiplicative noise,

$$\frac{\partial^2 X_t^\epsilon(\xi)}{\partial t^2} = \Delta X_t^\epsilon(\xi) + f(X_t^\epsilon(\xi), Y_t^\epsilon(\xi)) + \sigma_1(X_t^\epsilon(\xi)) \dot{W}_t^1(\xi), \quad (1.1)$$

$$\frac{\partial Y_t^\epsilon(\xi)}{\partial t} = \frac{1}{\epsilon} \Delta Y_t^\epsilon(\xi) + \frac{1}{\epsilon} g(X_t^\epsilon(\xi), Y_t^\epsilon(\xi)) + \frac{\sigma_2(X_t^\epsilon(\xi), Y_t^\epsilon(\xi))}{\sqrt{\epsilon}} \dot{W}_t^2(\xi), \quad (1.2)$$

$$X_t^\epsilon(\xi) = Y_t^\epsilon(\xi) = 0, (\xi, t) \in \partial D \times (0, T_0], \quad (1.3)$$

$$X_0^\epsilon(\xi) = X_0(\xi), Y_0^\epsilon(\xi) = Y_0(\xi), \frac{\partial X_t^\epsilon(\xi)}{\partial t} \Big|_{t=0} = \dot{X}_0(\xi), \xi \in D, \quad (1.4)$$

where the space variable  $\xi \in D$ , the time  $t \in [0, T_0]$ . Here the forcing noise  $W_t^1(\xi)$  and  $W_t^2(\xi)$  are mutually independent Wiener processes on a complete stochastic basis  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ , which will be specified later. Also, the precise conditions on  $f, g, \sigma_1$  and  $\sigma_2$  will be presented in the next section. The positive and small parameter  $\epsilon$  measures the ratio of the time scales between slow component  $X_t^\epsilon$  and fast component  $Y_t^\epsilon$ .

The system in form of (1.1)-(1.4) is an abstract model for a random vibration of a elastic string with external force on a large time scale. More generally, the nonlinear coupled wave-heat equations with fast and slow time scales may describe a thermoelastic wave propagation in a random medium [7], the interactions of fluid motion with other forms of waves [19, 28], wave phenomena which are heat generating or temperature related [18], magneto-elasticity [22] and biological problems [6, 9, 25].

We are often interested in the dynamical evolution of the slow component  $X_t^\epsilon$  as the scale parameter  $\epsilon$  goes to zero. Due to the time-scale separation, a simplified equation, which excludes fast component and approximates the dynamics of slow component, is highly desirable. Such a simplified equation can be obtained by the so-called averaging procedure. This result was proved in our previous paper [12] in the case of the noise is additive type.

Averaging principle is a powerful tool to analyze the asymptotic behavior for slow-fast dynamical systems. The averaging principle was first formulated by Bogoliubov[1] for deterministic differential equations. For its validity to stochastic differential equations (SDEs) with Gaussian noise, we mainly refer to the well known [13], the works of Freidlin and Wentzell [10, 11], Veretennikov [23, 24] and Kifer [15, 16, 17]. Further progress on averaging for stochastic dynamic systems with non-Gaussian noise is made in [26, 27]. Recently, there are increasing interests to extend the classical averaging about SDEs to the case of stochastic partial differential equations (SPDEs). In a series of works [5, 3, 4], averaging principle of SPDEs of reaction-diffusion type, with no explicit convergence rate being given, is studied.

Once the averaging principle is established, an important question arises as to how fast the original slow component will converge to the effective dynamics. In [2], when additive noise is included only in the fast component, explicit strong convergence rate of

order  $\frac{1}{2} - \varepsilon$  for arbitrary small  $\varepsilon > 0$  for averaging of stochastic parabolic equations is obtained. However, the order will be decreased to  $\frac{1}{5}$  if the noise also acts on the slow variables directly (see [2, Section 1]). These convergence rates can be compared with order  $\frac{1}{2}$  obtained for the finite dimensional stochastic dynamic systems [20, 14]. An interesting question thus occurs as to whether it is possible to get a order  $\frac{1}{2}$  for strong convergence in averaging of stochastic dynamic systems in infinite dimension, and, if yes, under which conditions. In this article, we present a positive answer to it by dealing with a hyperbolic-parabolic equation with multiplicative noise in form of (1.1)-(1.4). To be more precise, we will show the slow component  $X_t^\varepsilon$  can be approximated by the solution process of a reduced system, which enjoys strong convergence order  $\frac{1}{2}$  and is governed by a stochastic wave equation constructed by averaging the slow evolution with respect to its stationary measure. To construct the averaging system, a key point is to show the existence for an invariant measure with exponentially mixing property for the fast equation and this can be obtained by the same discussion as in [12], where a dissipative condition is needed. To provide an explicit error bounds on the difference between the solution of the original system and the solution of the averaged equation, we follow the general lines of the argument introduced in [20], but it is more delicate than in [20], as it involves the systems with spatial variable in infinite dimension. In order to obtain rate of convergence estimates we need to require more space and time regularity, differentiability with respect to the parameters of the solution for the coupled stochastic hyperbolic-parabolic system. Therefore in our setup we introduce additional derivable conditions on drift and diffusion coefficients.

The organization of this paper is as follows: In Section 2, we recall some basic concepts and results for later use. In Section 3, we prove the existence, uniqueness and energy identity for an abstract hyperbolic-parabolic equation. In Section 4, we study the ergodicity property for the fast motion of the system (1.1)-(1.4). In Section 5, some priori estimates is presented. Section 6 contains the main results of the paper as presented in Theorems 6.1. Finally, a necessary lemma is prove in the last section.

## 2 Preliminary

To rewrite the system (1.1)-(1.4) as the abstract evolution equation, we present some notations and recall some well-known facts for later use.

For a fixed domain  $D = (0, L)$ , let  $H$  be the Hilbert space  $L^2(D)$ , endowed with the usual scalar product  $(\cdot, \cdot)_H$  and the corresponding norm  $\|\cdot\|$ .

Let  $\{e_k(\xi)\}_{k \geq 1}$  denote the complete orthonormal system of eigenfunctions in  $H$  such that, for  $k = 1, 2, \dots$ ,

$$-\Delta e_k = \alpha_k e_k, \quad e_k(0) = e_k(L) = 0,$$

with  $0 < \alpha_1 \leq \alpha_2 \leq \dots \alpha_k \leq \dots$ . Here we would like to recall the fact that  $e_k(\xi) = \sin \frac{k\pi\xi}{L}$  and  $\alpha_k = -\frac{k^2\pi^2}{L^2}$  for  $k = 1, 2, \dots$ .

Let  $A$  be the realization in  $H$  of the Laplace operator  $\Delta$ , with zero Dirichlet boundary condition. For  $s \in \mathbb{R}$ , we introduce Hilbert space  $H_0^s = D((-A)^{s/2})$ , which equipped with

the inner scalar product

$$\langle u, v \rangle_s = \sum_{k=1}^{\infty} \alpha_k^s(u, e_k)_H (v, e_k)_H$$

and norm

$$\|u\|_s = \left\{ \sum_{k=1}^{\infty} \alpha_k^s(u, e_k)_H^2 \right\}^{\frac{1}{2}}$$

for  $u, v \in H_0^s$ .

Noticing that the Green function  $G(\xi, \zeta; t)$  for the deterministic equation  $(\partial/\partial t - A)X(t, \xi) = 0$  can be expressed as

$$G(\xi, \zeta, t) = \sum_{k=1}^{\infty} e^{-\alpha_k t} e_k(\xi) e_k(\zeta).$$

Recall that the associated Green's operator defined by, for any  $h(\xi) \in H$ ,

$$G_t h(\xi) = \int_D G(\xi, \zeta, t) h(\zeta) d\zeta = \sum_{k=1}^{\infty} e^{-\alpha_k t} e_k(\xi) (e_k, h)_H.$$

It is straightforward to check that  $\{G_t\}_{t \geq 0}$  are contractive semigroups on  $H$ . For the deterministic wave equation  $(\partial^2/\partial t^2 - A)Y(t, \xi) = 0$ , its Green's function is given by

$$S(\xi, \zeta, t) = \sum_{k=1}^{\infty} \frac{\sin\{\sqrt{\alpha_k} t\}}{\sqrt{\alpha_k}} e_k(\xi) e_k(\zeta),$$

It is easy to shown that the above series converge in  $L^2(D \times D)$  and the associated Green's operator is defined by, for any  $h(\xi) \in H$ ,

$$S_t h(\xi) = \int_D S(\xi, \zeta, t) h(\zeta) d\zeta = \sum_{k=1}^{\infty} \frac{\sin\{\sqrt{\alpha_k} t\}}{\sqrt{\alpha_k}} e_k(\xi) (e_k, h)_H.$$

For Green operator  $S_t$ , it is easy to derive the following results (see [8]):

**Proposition 2.1** *Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a complete stochastic basis and let  $\phi_t(\cdot) \in L^2(\Omega \times (0, T); H)$  be a  $\mathcal{F}_t$ -adapted process which satisfies  $\mathbb{E} \int_0^T \|\phi_t(\cdot)\|^2 dt < \infty$ . Then  $\nu_t(\cdot) = \int_0^t S_{t-s} \phi_s(\cdot) ds$  is a continuous,  $\mathcal{F}_t$ -adapted  $H_0^1$ -valued process and its time derivative  $\dot{\nu}_t(\cdot) = \frac{\partial}{\partial t} \nu_t(\cdot)$  is a continuous  $H$ -valued process such that*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\nu_t(\cdot)\|_1^2 \leq T \mathbb{E} \int_0^T \|\phi_s(\cdot)\|^2 ds \quad (2.1)$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\dot{\nu}_t(\cdot)\|^2 \leq T \mathbb{E} \int_0^T \|\phi_s(\cdot)\|^2 ds. \quad (2.2)$$

Next we recall the definition of the Wiener process in infinite space. For more details, see [8] or [21]. Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a complete stochastic basis. Let  $W_t(\cdot)$  be an  $H$ -valued

Wiener process with mean zero and covariance

$$\mathbb{E}(W_t(\xi)W_s(\eta)) = \min(s, t)r(\xi, \eta), \quad s, t \geq 0, \quad \xi, \eta \in D,$$

where the covariance function  $r(\xi, \eta)$  is positive and bounded such that

$$r_0 = \sup_{\xi \in D} r(\xi, \xi) < \infty. \quad (2.3)$$

The associated covariance operator  $\mathcal{R} : H \rightarrow H$  with symmetric kernel  $r(\xi, \eta)$  is defined by

$$(\mathcal{R}\varphi)(\xi) = \int_D r(\xi, \eta)\varphi(\eta)d\eta, \quad \xi \in D.$$

With condition (2.3), we know that  $\mathcal{R}$  is a Hilbert-Schmidt operator. Therefore,  $W_t$  is often called  $\mathcal{R}$ -Wiener process in  $H$ .

**Proposition 2.2** *Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a complete stochastic basis and let  $\phi_t(\cdot) \in L^2(\Omega \times (0, T); H)$  be a  $\mathcal{F}_t$ -adapted process which satisfies  $\mathbb{E} \int_0^T \|\phi_t(\cdot)\|^2 dt < \infty$ . Then the stochastic integral*

$$I_t = \int_0^t \phi_s(\cdot) dW_s, \quad t \in [0, T],$$

*is well defined as continuous square integrable martingale in  $H$  with Itô isometry*

$$\mathbb{E}\|I_t\|^2 = \mathbb{E} \int_0^t \|\phi_s\|_{\mathcal{R}}^2 ds,$$

*here  $\|\phi_t\|_{\mathcal{R}}^2 = \int_D r(\xi, \xi)\phi_t^2(\xi)d\xi$ .*

**Proposition 2.3** *Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  be a complete stochastic basis and let  $\phi_t(\cdot) \in L^2(\Omega \times (0, T); H)$  be a  $\mathcal{F}_t$ -adapted process which satisfies  $\mathbb{E} \int_0^T \|\phi_t(\cdot)\|^2 dt < \infty$ . Then*

$$\mu_t(\cdot) = \int_0^t S_{t-s}\phi_s(\cdot) dW_s, \quad t \in [0, T]$$

*is a continuous adapted  $H_0^1$ -valued process for  $t \in [0, T]$ , and its derivative  $\dot{\mu}_t(\cdot) = \frac{\partial}{\partial t}\mu_t(\cdot)$  is a continuous process in  $H$  (see Section 5.2 in [8] for details). Moreover, the following inequalities hold:*

$$\mathbb{E}\|\mu_t(\cdot)\|_1^2 \leq \mathbb{E} \int_0^t \|\phi_s(\cdot)\|_{\mathcal{R}}^2 ds, \quad (2.4)$$

$$\mathbb{E}\|\dot{\mu}_t(\cdot)\|^2 \leq \mathbb{E} \int_0^t \|\phi_s(\cdot)\|_{\mathcal{R}}^2 ds. \quad (2.5)$$

A natural way to give a rigorous meaning to Eq. (1.1)-(1.4) is in terms of the following

integral equations:

$$X_t^\epsilon = S_t' X_0 + S_t \dot{X}_0 + \int_0^t S_{t-s} f(X_s^\epsilon, Y_s^\epsilon) ds + \int_0^t S_{t-s} \sigma_1(X_s^\epsilon) dW_s^1, \quad (2.6)$$

$$\begin{aligned} Y_t^\epsilon &= G_{t/\epsilon} Y_0 + \frac{1}{\epsilon} \int_0^t G_{(t-s)/\epsilon} g(X_s^\epsilon, Y_s^\epsilon) ds \\ &\quad + \frac{1}{\sqrt{\epsilon}} \int_0^t G_{(t-s)/\epsilon} \sigma_2(X_s^\epsilon, Y_s^\epsilon) dW_s^2, \end{aligned} \quad (2.7)$$

where  $S_t' = \frac{d}{dt} S_t$  is the derived Green's operator with integral kernel

$$K'(\xi, \zeta, t) = \sum_{k=1}^{\infty} \cos\{\sqrt{\alpha_k} t\} e_k(\xi) e_k(\zeta).$$

As a solution to Eq. (1.1)-(1.4), we take the so-called mild solution.

**Definition 2.1** *If  $(X_t^\epsilon, Y_t^\epsilon)$  is an adapted process over  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  such that  $\mathbb{P}$ -a.s. the integral Eq. (2.6)-(2.7) is hold true for all  $t > 0$ , we say that it is a mild solution for Eq. (1.1)-(1.4).*

Let us introduce the following set of additional assumptions.

(A1) For the mapping  $f : H \times H \rightarrow H$ , we require that there exists a constant  $L_f$  such that for any  $x, y, h, k \in H$  its directional derivatives are well-defined and satisfy

$$\begin{aligned} \|D_x f(x, y) \cdot h\| &\leq L_f \|h\|, \\ \|D_y f(x, y) \cdot h\| &\leq L_f \|h\|, \\ \|D_{xx}^2 f(x, y) \cdot (h, k)\| &\leq L_f \|h\| \cdot \|k\|, \\ \|D_{yy}^2 f(x, y) \cdot (h, k)\| &\leq L_f \|h\| \cdot \|k\|, \\ \|D_{xy}^2 f(x, y) \cdot (h, k)\| &\leq L_f \|h\| \cdot \|k\|, \\ \|D_{yx}^2 f(x, y) \cdot (h, k)\| &\leq L_f \|h\| \cdot \|k\|. \end{aligned}$$

(A2) Assume the mapping  $\sigma_1 : H \rightarrow H$  satisfies the global Lipschitz condition and the sublinear growth.

(A3) For the mapping  $g : H \times H \rightarrow H$  and  $\sigma_2 : H \times H \rightarrow H$ , we assume the regularity conditions presented in the assumption (A1) also hold for  $g$  and  $\sigma_2$  with the constant  $L_f$  replaced by  $L_g$  and  $L_{\sigma_2}$ , respectively.

(A4) For  $i = 1, 2$ , the covariance function  $r_i(\xi, \eta)$  with respect to  $H$ -valued  $\mathcal{R}_i$ -Wiener process  $W_t^i$  is bounded by constant  $r_i$ , that is,

$$0 < r_i(\xi, \eta) \leq r_i, \quad \xi, \eta \in D.$$

(A5) Assume the fast motion equation satisfies the strong dissipative condition, that is

$$\kappa := 2\alpha_1 - 2L_g - r_2 L_{\sigma_2}^2 > 0. \quad (2.8)$$

**Remark 2.1** *With Assumption (A1) and (A3), it is clear that  $f, g$  and  $\sigma_2$  are Lipschitz mapping from  $H \times H$  to  $H$ , and hence have linear growth bounds.*

**Remark 2.2** *For the convenience of notations, the symbols  $C$  with or without subscripts will denote a positive constant that is unimportant and may have different values from one line to another one in the sequel.*

### 3 Existence, uniqueness and energy equality

In this section we introduce the existence and uniqueness for Eqs. (2.6)-(2.7). This is a standard approach based on successive approximations. Let  $U_t$  and  $V_t$  be  $\mathcal{F}_t$ -adapted processes in  $H$  such that

$$\mathbb{E} \int_0^T (\|U_s\|^2 + \|\tilde{U}_s\|^2 + \|V_s\|^2 + \|\tilde{V}_s\|^2) ds < \infty. \quad (3.1)$$

Now, for fixed  $x_0 \in H_0^1, \dot{x}_0, y_0 \in H$  we first consider the integrals

$$X_t = S'_t x_0 + S_t \dot{x}_0 + \int_0^t S_{t-s} U_s ds + \int_0^t S_{t-s} \tilde{U}_s dW_s^1 \quad (3.2)$$

and

$$Y_t = G_t y_0 + \int_0^t G_{t-s} V_s ds + \int_0^t G_{t-s} \tilde{V}_s dW_s^2. \quad (3.3)$$

They are respectively mild solutions of the linear equations with additive noise

$$\frac{d^2}{dt^2} X_t = AX_t + U_t + \tilde{U}_t \dot{W}_t^1, X_0 = x_0, \frac{dX_t}{dt} \Big|_{t=0} = \dot{x}_0$$

and

$$\frac{d}{dt} Y_t = AY_t + V_t + \tilde{V}_t \dot{W}_t^2, Y_0 = y_0.$$

By Chow [8, Theorem 3.5, Chapter 5], the linear problem (3.2) has a unique solution

$$X_t \in L^2(\Omega; C([0, T_0]; H_0^1))$$

with regularity  $\dot{X}_t = \frac{d}{dt} X_t \in L^2(\Omega; C([0, T_0]; H))$ . Moreover the energy equality holds

$$\begin{aligned} \|\dot{X}_t\|^2 + \|X_t\|_1^2 &= \|\dot{x}_0\|^2 + \|x_0\|_1^2 + 2 \int_0^t (\dot{X}_s, U_s)_H ds + 2 \int_0^t (X_s, \tilde{U}_s dW_s^1)_H \\ &\quad + \int_0^t \|\tilde{U}_s\|_{\mathcal{R}_1}^2 ds, \quad a.s.. \end{aligned} \quad (3.4)$$

By Chow [8, Theorem 5.3, Chapter 3], the linear problem (3.3) has a unique solution which is a process in  $H_0^1$  with continuous path in  $H$  such that the energy equality holds true:

$$\|Y_t\|^2 = \|y_0\|^2 + 2 \int_0^t \langle AY_s, Y_s \rangle ds + 2 \int_0^t (Y_s, V_s)_H ds + 2 \int_0^t (Y_s, \tilde{V}_s dW_s^2)_H$$

$$+ \int_0^t \|\tilde{V}_s\|_{\mathcal{R}_2}^2 ds, \quad a.s.. \quad (3.5)$$

Now, for fixed  $X_0 \in H_0^1, \dot{X}_0, Y_0 \in H$  we consider the nonlinear problem

$$X_t = S'_t X_0 + S_t \dot{X}_0 + \int_0^t S_{t-s} f(X_s, Y_s) ds + \int_0^t S_{t-s} \sigma_1(X_s) dW_s^1, \quad (3.6)$$

$$Y_t = G_t Y_0 + \int_0^t G_{t-s} g(X_s, Y_s) ds + \int_0^t G_{t-s} \sigma_2(X_s, Y_s) dW_s^2. \quad (3.7)$$

**Theorem 3.1** *Assume that assumption (A1)-(A4) are satisfied. Given  $X_0 \in H_0^1, \dot{X}_0, Y_0 \in H$ , then the system (3.6)-(3.7) has a unique mild solution*

$$(X_t, Y_t) \in L^2(\Omega; C([0, T_0]; H_0^1)) \times L^2(\Omega; C([0, T_0]; H))$$

with  $\dot{X}_t = \frac{d}{dt} X_t \in L^2(\Omega; C([0, T_0]; H))$ . Moreover, the solution enjoys the energy equations

$$\begin{aligned} \|\dot{X}_t\|^2 + \|X_t\|_1^2 &= \|\dot{X}_0\|^2 + \|X_0\|_1^2 + 2 \int_0^t (\dot{X}_s, f(X_s, Y_s))_H ds \\ &\quad + 2 \int_0^t (X_s, \sigma_1(X_s) dW_s^1)_H + \int_0^t \|\sigma_1(X_s)\|_{\mathcal{R}_1}^2 ds, \quad a.s., \end{aligned} \quad (3.8)$$

$$\begin{aligned} \|Y_t\|^2 &= \|Y_0\|^2 + 2 \int_0^t \langle AY_s, Y_s \rangle ds + 2 \int_0^t (Y_s, g(X_s, Y_s))_H ds \\ &\quad + 2 \int_0^t (Y_s, \sigma_2(X_s, Y_s) dW_s^2)_H + \int_0^t \|\sigma_2(X_s, Y_s)\|_{\mathcal{R}_2}^2 ds, \quad a.s.. \end{aligned} \quad (3.9)$$

Proof: We will verify the existence by successive approximations. Let

$$\begin{aligned} X_t^0 &= X_0 + \int_0^t \dot{X}_0 ds, \\ Y_t^0 &= Y_0, \quad t \geq 0. \end{aligned}$$

For  $n \geq 1$ , let  $(X_t^n, Y_t^n)$  be the unique solution to the linear equation

$$\begin{aligned} X_t^n &= S'_t X_0 + S_t \dot{X}_0 + \int_0^t S_{t-s} f(X_s^{n-1}, Y_s^{n-1}) ds \\ &\quad + \int_0^t S_{t-s} \sigma_1(X_s^{n-1}, Y_s^{n-1}) dW_s^1, \end{aligned} \quad (3.10)$$

$$\begin{aligned} Y_t^n &= G_t Y_0 + \int_0^t G_{t-s} g(X_s^{n-1}, Y_s^{n-1}) ds \\ &\quad + \int_0^t G_{t-s} \sigma_2(X_s^{n-1}, Y_s^{n-1}) dW_s^2. \end{aligned} \quad (3.11)$$

Note that the solution  $(X_t^n, Y_t^n)$  of the above equation exists according to former discussion



for linear system. Additional, the time derivative of  $X_t^n$  satisfies

$$\begin{aligned}\dot{X}_t^n &= S_t'' X_0 + S_t' \dot{X}_0 + \int_0^t S_{t-s}' f(X_s^{n-1}, Y_s^{n-1}) ds \\ &\quad + \int_0^t S_{t-s}' \sigma_1(X_s^{n-1}, Y_s^{n-1}) dW_s^1,\end{aligned}$$

where  $S_t''$  denotes the second derivative of Green's operator with integral kernel

$$K''(\xi, \zeta, t) = - \sum_{k=1}^{\infty} \sqrt{\alpha_k} \sin\{\sqrt{\alpha_k} t\} e_k(\xi) e_k(\zeta).$$

We are going to show that  $\{(X_t^n, Y_t^n)\}_{n \geq 1}$  forms a Cauchy sequence. For any  $t \in [0, T_0]$ , the energy equality (3.4) yields

$$\begin{aligned}&\|\dot{X}_t^{n+1} - \dot{X}_t^n\|^2 + \|X_t^{n+1} - X_t^n\|_1^2 \\ &= 2 \int_0^t (\dot{X}_s^{n+1} - \dot{X}_s^n, f(X_s^n, Y_s^n) - f(X_s^{n-1}, Y_s^{n-1}))_H ds \\ &\quad + 2 \int_0^t (X_s^{n+1} - X_s^n, (\sigma_1(X_s^n) - \sigma_1(X_s^{n-1}))) dW_s^1 \\ &\quad + \int_0^t \|\sigma_1(X_s^n) - \sigma_1(X_s^{n-1})\|_{\mathcal{R}_1}^2 ds\end{aligned}$$

From this equality together with the assumptions (A1), (A2) and (A4), we can deduce that

$$\begin{aligned}&\mathbb{E} \sup_{0 \leq s \leq t} [\|\dot{X}_s^{n+1} - \dot{X}_s^n\|^2 + \|X_s^{n+1} - X_s^n\|_1^2] \\ &\leq C \mathbb{E} \left[ \int_0^t \|\dot{X}_s^{n+1} - \dot{X}_s^n\|^2 ds + \int_0^t \|X_s^{n+1} - X_s^n\|^2 ds \right. \\ &\quad \left. + \int_0^t \|X_s^n - X_s^{n-1}\|^2 ds + \int_0^t \|Y_s^n - Y_s^{n-1}\|^2 ds \right] \quad (3.12)\end{aligned}$$

Notice that for  $s \in [0, t]$ , we have

$$\begin{aligned}\dot{X}_s^{n+1} - \dot{X}_s^n &= \int_0^s S_{s-r}' (f(X_r^n, Y_r^n) - f(X_r^{n-1}, Y_r^{n-1})) dr \\ &\quad + \int_0^s S_{s-r}' (\sigma_1(X_r^n) - \sigma_1(X_r^{n-1})) dW_r^1\end{aligned}$$

and then, due to the contractive property of  $\{S_t'\}_{t \geq 0}$  on  $H$ , by Hölder's inequality we obtain

$$\begin{aligned}\mathbb{E} \|\dot{X}_s^{n+1} - \dot{X}_s^n\|^2 &\leq s \int_0^s \mathbb{E} \|f(X_r^n, Y_r^n) - f(X_r^{n-1}, Y_r^{n-1})\|^2 dr \\ &\quad + C \int_0^s \mathbb{E} \|\sigma_1(X_r^n) - \sigma_1(X_r^{n-1})\|^2 dr \\ &\leq sC \int_0^s \|X_r^n - X_r^{n-1}\|^2 + \|Y_r^n - Y_r^{n-1}\|^2 dr.\end{aligned}$$

This means

$$\mathbb{E} \int_0^t \|\dot{X}_s^{n+1} - \dot{X}_s^n\|^2 ds \leq C_{T_0} \int_0^t (\mathbb{E} \|X_r^n - X_r^{n-1}\|^2 + \mathbb{E} \|Y_r^n - Y_r^{n-1}\|^2) dr.$$

Together with (3.12), this yields

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq t} [\|\dot{X}_s^{n+1} - \dot{X}_s^n\|^2 + \|X_s^{n+1} - X_s^n\|_1^2] \\ & \leq C \mathbb{E} \int_0^t [\|X_s^{n+1} - X_s^n\|^2 + \|X_s^n - X_s^{n-1}\|^2 \\ & \quad + \|Y_s^n - Y_s^{n-1}\|^2] ds \\ & \leq C \mathbb{E} \int_0^t [\|X_s^{n+1} - X_s^n\|_1^2 + \|X_s^n - X_s^{n-1}\|_1^2 \\ & \quad + \|Y_s^n - Y_s^{n-1}\|^2] ds \end{aligned} \quad (3.13)$$

By energy equality (3.5), assumption (A3) and the fact  $\langle A(Y_s^{n+1} - Y_s^n), Y_s^{n+1} - Y_s^n \rangle \leq 0$ , we obtain

$$\begin{aligned} \|Y_t^{n+1} - Y_t^n\|^2 & \leq 2 \int_0^t (Y_s^{n+1} - Y_s^n, g(X_s^n, Y_s^n) - g(X_s^{n-1}, Y_s^{n-1}))_H ds \\ & \quad + 2 \int_0^t (Y_s^{n+1} - Y_s^n, (\sigma_2(X_s^n, Y_s^n) - \sigma_2(X_s^{n-1}, Y_s^{n-1})) dW_s^2)_H \\ & \quad + \int_0^t \|\sigma_2(X_s^n, Y_s^n) - \sigma_2(X_s^{n-1}, Y_s^{n-1})\|_{\mathcal{R}_2}^2 ds \\ & \leq C \int_0^t [\|Y_s^{n+1} - Y_s^n\|^2 + \|X_s^n - X_s^{n-1}\|^2 + \|Y_s^n - Y_s^{n-1}\|^2] ds \\ & \leq C \int_0^t [\|Y_s^{n+1} - Y_s^n\|^2 + \|X_s^n - X_s^{n-1}\|_1^2 + \|Y_s^n - Y_s^{n-1}\|^2] ds. \end{aligned}$$

According to (3.13), this implies that we have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq r \leq t} [\|X_r^{n+1} - X_r^n\|_1^2 + \|Y_r^{n+1} - Y_r^n\|^2] \\ & \leq C \mathbb{E} \left[ \int_0^t (\|X_s^{n+1} - X_s^n\|_1^2 + \|Y_s^{n+1} - Y_s^n\|^2) ds \right] \\ & \quad + C \mathbb{E} \left[ \int_0^t (\|X_s^n - X_s^{n-1}\|_1^2 + \|Y_s^n - Y_s^{n-1}\|^2) ds \right]. \end{aligned} \quad (3.14)$$

Iterating Eq. (3.14) we obtain that

$$\mathbb{E} \sup_{0 \leq s \leq T_0} [\|X_s^{n+1} - X_s^n\|_1^2 + \|Y_s^{n+1} - Y_s^n\|^2] \leq C \frac{(C_{T_0})^n}{n!}.$$

This implies that there exists

$$(X_t, Y_t) \in L^2(\Omega; C([0, T_0]; H_0^1)) \times L^2(\Omega; C([0, T_0]; H))$$

such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \sup_{0 \leq t \leq T_0} [\|X_t^n - X_t\|_1^2 + \|Y_t^n - Y_t\|^2] = 0.$$

In view of inequality (3.13) we see that the sequence  $\{\dot{X}_t^n\}_{n \geq 1}$  also converges to  $\dot{X}_t$  in  $L^2(\Omega; C([0, T_0]; H))$ . Letting  $n \rightarrow \infty$  in Eq. (3.10) and Eq. (3.11), it is seen that  $(X_t, Y_t)$  is a solution to Eqs. (3.6)-(3.7). The uniqueness is a directive consequence of energy equalities and Gronwall's inequality. To verify the energy equalities, one has the following convergence in  $L^2(\mathbb{P})$  as  $n \rightarrow \infty$  for all  $0 \leq t \leq T_0$ :

$$\dot{X}_t^n \rightarrow \dot{X}_t, \quad (3.15)$$

$$X_t^n \rightarrow X_t, \quad (3.16)$$

and

$$\int_0^t (X_s^n, \sigma_1(X_s^n) dW_s^1)_H ds \rightarrow \int_0^t (X_s, \sigma_1(X_s) dW_s^1)_H ds, \quad (3.17)$$

$$\int_0^t \|\sigma_1(X_s^n)\|_{\mathcal{R}_1}^2 ds \rightarrow \int_0^t \|\sigma_1(X_s)\|_{\mathcal{R}_1}^2 ds, \quad (3.18)$$

$$\int_0^t (\dot{X}_s^n, f(X_s^n, Y_s^n))_H ds \rightarrow \int_0^t (\dot{X}_s, f(X_s, Y_s))_H ds \quad (3.19)$$

in mean for all  $0 \leq t \leq T_0$ . Then by taking a subsequence converging  $\mathbb{P}$ -a.s. for Eqs. (3.15)-(3.19), one can obtain the energy equality given by Eq. (3.8). By a similar calculation we can get the energy equality (3.9).  $\blacksquare$

## 4 Ergodicty for frozen equation

For fixed  $x \in H$  consider the problem associate to fast motion with frozen slow component.

$$\frac{\partial Y_t(\xi)}{\partial t} = \Delta Y_t(\xi) + g(x, Y_t(\xi)) + \sigma_2(x, Y_t(\xi)) \dot{W}_t^2, \quad (4.1)$$

$$Y_t(\xi) = 0, (\xi, t) \in \partial D \times [0, \infty), \quad (4.2)$$

$$Y_0(\xi) = y, \quad (4.3)$$

By arguing as before, for any fixed slow component  $x \in H$  and any initial data  $y \in H$ , system (4.1)-(4.3) has a unique mild solution denoted by  $Y_t^{x,y}$ . The energy equality (3.9) reads

$$\begin{aligned} \|Y_t^{x,y}\|^2 &= \|y\|^2 + 2 \int_0^t \langle AY_s^{x,y}, Y_s^{x,y} \rangle ds + 2 \int_0^t (g(x, Y_s^{x,y}), Y_s^{x,y})_H ds \\ &\quad + 2 \int_0^t (Y_s^{x,y}, \sigma_2(x, Y_s^{x,y}) dW_s^2)_H + \int_0^t \|\sigma_2(x, Y_s^{x,y})\|_{\mathcal{R}_2}^2 ds, \quad a.s., \end{aligned}$$

which implies the differential form

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \|Y_t^{x,y}\|^2 &= 2\mathbb{E} \langle AY_t^{x,y}, Y_t^{x,y} \rangle + 2\mathbb{E} (g(x, Y_t^{x,y}), Y_t^{x,y})_H \\ &\quad + \mathbb{E} \|\sigma_2(x, Y_t^{x,y})\|_{\mathcal{R}_2}^2, \end{aligned} \quad (4.4)$$

and then, thanks to the Poincaré inequality and the Lipschitz continuity of  $g$  and  $\sigma_2$ , we obtain

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \|Y_t^{x,y}\|^2 &\leq -2\alpha_1 \mathbb{E} \|Y_t^{x,y}\|^2 + 2\mathbb{E} (g(x, Y_t^{x,y}) - g(x, 0), Y_t^{x,y})_H \\ &\quad + 2\mathbb{E} (g(x, 0), Y_t^{x,y})_H + r_2 \|\sigma_2(x, Y_t^{x,y}) - \sigma_2(x, 0)\|^2 \\ &\quad + r_2 \|\sigma_2(x, 0)\|^2 + 2r_2 (\sigma_2(x, Y_t^{x,y}) - \sigma_2(x, 0), \sigma_2(x, 0))_H \\ &\leq -(2\alpha_1 - 2L_g - r_2 L_{\sigma_2}^2 - \rho) \mathbb{E} \|Y_t^{x,y}\|^2 + C_\rho (1 + \|x\|^2) \end{aligned}$$

with positive  $\rho$  and  $C_\rho$  independent of  $x$  and  $Y_t^{x,y}$ , where we have used the Young inequality in the form  $|a_1 a_2| \leq \rho |a_1|^2 + C_\rho |a_2|^2$  for  $\rho > 0$  at the second step. Hence taking (2.8) into account and choosing  $\rho$  small enough we can find  $C_1, C_2 > 0$  such that

$$\frac{d}{dt} \mathbb{E} \|Y_t^{x,y}\|^2 \leq -C_1 \mathbb{E} \|Y_t^{x,y}\|^2 + C_2 (1 + \|x\|^2),$$

which, by Gronwall's inequality, implies that

$$\mathbb{E} \|Y_t^{x,y}\|^2 \leq C (e^{-ct} \|y\|^2 + \|x\|^2 + 1), \quad t > 0 \quad (4.5)$$

for some constants  $c, C > 0$ . This implies that for any  $x \in H$ , there exists an invariant measure  $\mu^x$  for the Markov semigroup  $P_t^x$  associated with system (4.1)-(4.3) in  $H$  such that

$$\int_H P_t^x \psi d\mu^x = \int_H \psi d\mu^x, \quad t \geq 0$$

for any  $\psi \in \mathcal{B}_b(H)$  the space of bounded functions on  $H$  (for a proof, see, e.g., [4], Section 2.1). Then by repeating the standard argument as in the proof of Proposition 4.2 in [5], the invariant measure has finite 2-moments:

$$\int_H \|y\|^2 \mu^x(dy) \leq C(1 + \|x\|^2). \quad (4.6)$$

Let  $Y_t^{x,y'}$  be the solution of system(4.1)-(4.3) with initial value  $Y_0 = y'$ , the energy equality implies that for any  $t \geq 0$ ,

$$\mathbb{E} \|Y_t^{x,y} - Y_t^{x,y'}\|^2 \leq C \|y - y'\|^2 e^{-ct}$$

with  $C, c > 0$ , which implies that  $\mu^x$  is the unique invariant measure for  $P_t^x$ . Then, according to the invariant property of  $\mu^x$ , (4.6) and hypothesis (A1), we have

$$\left\| \mathbb{E} f(x, Y_t^{x,y}) - \int_H f(x, z) \mu^x(dz) \right\|^2 = \left\| \int_H \mathbb{E} (f(x, Y_t^{x,y}) - f(x, Y_t^{x,z})) \mu^x(dz) \right\|^2$$

$$\begin{aligned}
 &\leq C \int_H \mathbb{E} \|Y_t^{x,y} - Y_t^{x,z}\|^2 \mu^x(dz) \\
 &\leq C e^{-ct} \int_H \|y - z\|^2 \mu^x(dz) \\
 &\leq C e^{-ct} (1 + \|x\|^2 + \|y\|^2).
 \end{aligned} \tag{4.7}$$

## 5 A priori bounds for the slow-fast system

We now prove the following estimate about the solution processes  $X_t^\epsilon$  and  $Y_t^\epsilon$  of the system (2.6)-(2.7).

**Lemma 5.1** *Assume that  $X_0 \in H_0^1$ ,  $\dot{X}_0 \in H$ ,  $Y_0 \in H$  and (A1)-(A5) hold, then there exists a constant  $C_{T_0} > 0$  such that*

$$\sup_{\substack{\epsilon > 0, \\ 0 \leq t \leq T_0}} \mathbb{E} (\|X_t^\epsilon\|_1^2 + \|\dot{X}_t^\epsilon\|^2) \leq C_{T_0} \left( 1 + \|Y_0\|^2 + \|X_0\|_1^2 + \|\dot{X}_0\|^2 \right) \tag{5.1}$$

and

$$\sup_{\substack{\epsilon > 0, \\ 0 \leq t \leq T_0}} \mathbb{E} \|Y_t^\epsilon\|^2 \leq C_{T_0} \left( 1 + \|Y_0\|^2 + \|X_0\|_1^2 + \|\dot{X}_0\|^2 \right) \tag{5.2}$$

Proof: By the energy equality (3.8) and sublinear growth condition of  $f$ , we have

$$\begin{aligned}
 \mathbb{E} \left( \|\dot{X}_t^\epsilon\|^2 + \|X_t^\epsilon\|_1^2 \right) &= \|\dot{X}_0\|^2 + \|X_0\|_1^2 + 2\mathbb{E} \int_0^t (\dot{X}_s^\epsilon, f(X_s^\epsilon, Y_s^\epsilon))_H ds \\
 &\quad + \int_0^t \mathbb{E} \|\sigma_1(X_s^\epsilon)\|_{\mathcal{R}_1}^2 ds \\
 &\leq \|\dot{X}_0\|^2 + \|X_0\|_1^2 + C \int_0^t \mathbb{E} \left( \|\dot{X}_s^\epsilon\|^2 + \|X_s^\epsilon\|_1^2 \right) ds \\
 &\quad + C \int_0^t \mathbb{E} (1 + \|Y_s^\epsilon\|^2) ds,
 \end{aligned}$$

so that

$$\begin{aligned}
 \mathbb{E} \left( \|\dot{X}_t^\epsilon\|^2 + \|X_t^\epsilon\|_1^2 \right) &\leq e^{Ct} \left( \|\dot{X}_0\|^2 + \|X_0\|_1^2 \right) \\
 &\quad + C \int_0^t e^{C(t-s)} (1 + \mathbb{E} \|Y_s^\epsilon\|^2) ds.
 \end{aligned} \tag{5.3}$$

Thanks to energy equality (3.9) and (A5), we have

$$\begin{aligned}
 \frac{d}{dt} \mathbb{E} \|Y_t^\epsilon\| &= \frac{2}{\epsilon} \mathbb{E} \langle AY_t^\epsilon, Y_t^\epsilon \rangle + \frac{2}{\epsilon} \mathbb{E} (g(X_t^\epsilon, Y_t^\epsilon), Y_t^\epsilon)_H + \frac{1}{\epsilon} \mathbb{E} \|\sigma_2(X_t^\epsilon, Y_t^\epsilon)\|_{\mathcal{R}_2}^2 ds \\
 &\leq -\frac{C_1}{\epsilon} \mathbb{E} \|Y_t^\epsilon\|^2 + \frac{C_2}{\epsilon} (1 + \mathbb{E} \|X_t^\epsilon\|^2),
 \end{aligned}$$

and hence

$$\mathbb{E}\|Y_t^\epsilon\| \leq e^{C_2 t} \|Y_0\|^2 + \frac{C_2}{\epsilon} \int_0^t e^{-\frac{C_1}{\epsilon}(t-s)} (1 + \mathbb{E}\|X_s^\epsilon\|^2) ds.$$

According to (5.3), we obtain

$$\begin{aligned} \mathbb{E}\|Y_t^\epsilon\| &\leq C_{T_0} \left(1 + \|Y_0\|^2 + \|\dot{X}_0\|^2 + \|X_0\|_1^2\right) \\ &\quad + \frac{C_{T_0}}{\epsilon} \int_0^t e^{-\frac{C_1}{\epsilon}(t-s)} \int_0^s \mathbb{E}\|Y_r^\epsilon\|^2 dr ds. \end{aligned}$$

By change of variables, this yields

$$\begin{aligned} \mathbb{E}\|Y_t^\epsilon\| &\leq C_{T_0} \left(1 + \|Y_0\|^2 + \|\dot{X}_0\|^2 + \|X_0\|_1^2\right) \\ &\quad + C_{T_0} \int_0^t \mathbb{E}\|Y_r^\epsilon\|^2 \left[ \int_0^{\frac{t-r}{\epsilon}} e^{-C_1 v} dv \right] dr \\ &\leq C_{T_0} \left(1 + \|Y_0\|^2 + \|\dot{X}_0\|^2 + \|X_0\|_1^2\right) \\ &\quad + C_{T_0} \int_0^t \mathbb{E}\|Y_r^\epsilon\|^2 dr. \end{aligned}$$

Hence, by Gronwall's inequality we get

$$\mathbb{E}\|Y_t^\epsilon\| \leq C_{T_0} \left(1 + \|Y_0\|^2 + \|\dot{X}_0\|^2 + \|X_0\|_1^2\right),$$

which, gives the estimate (5.2). By replacing the estimate above in (5.3) and using the Gronwall inequality again, we obtain the first estimate (5.1).  $\blacksquare$

We now provide a regularity estimates of  $X_t^\epsilon$  in the time variable.

**Lemma 5.2** *Suppose that conditions in Lemma 5.1 hold. For any  $X_0 \in H_0^1, \dot{X}_0 \in H, Y_0 \in H$ , there exists a constant  $C > 0$  such that for all  $t > 0$  and  $h \in (0, 1)$ ,*

$$\sup_{\epsilon > 0} \mathbb{E}\|X_{t+h}^\epsilon - X_t^\epsilon\|^2 \leq C(\|X_0\|_1^2 + \|\dot{X}_0\|^2 + \|Y_0\|^2)h^2. \quad (5.4)$$

Proof: Clearly, we have

$$\begin{aligned} \mathbb{E}\|X_{t+h}^\epsilon - X_t^\epsilon\|^2 &= \mathbb{E}\left\|\int_t^{t+h} \dot{X}_s^\epsilon ds\right\|^2 \\ &\leq h \int_t^{t+h} \mathbb{E}\|\dot{X}_s^\epsilon\|^2 ds, \end{aligned}$$

so that, by (5.1),

$$\mathbb{E}\|X_{t+h}^\epsilon - X_t^\epsilon\|^2 \leq Ch^2.$$

$\blacksquare$

Next, we introduce an auxiliary process  $(\hat{X}_t^\epsilon, \hat{Y}_t^\epsilon) \in H_0^1 \times H$ . Fix a positive number  $\delta < 1$  and do a partition of time interval  $[0, T_0]$  of size  $\delta$ . We construct a process  $\hat{Y}_t^\epsilon$ , with initial datum  $\hat{Y}_0^\epsilon = Y_0$ , by means of the equations

$$d\hat{Y}_t^\epsilon = \frac{1}{\epsilon} A \hat{Y}_t^\epsilon dt + \frac{1}{\epsilon} g(X_{k\delta}^\epsilon, \hat{Y}_t^\epsilon) dt + \frac{1}{\sqrt{\epsilon}} \sigma_2(X_{k\delta}^\epsilon, \hat{Y}_t^\epsilon) dW_t^2$$

for  $t \in [k\delta, \min\{(k+1)\delta, T_0\})$ ,  $k \geq 0$ , where  $X_{k\delta}^\epsilon$  is the slow solution processes at time  $k\delta$ . For the left end of each subinterval we set

$$\hat{Y}_{(k+1)\delta}^\epsilon = \lim_{t \rightarrow (k+1)\delta^-} \hat{Y}_t^\epsilon.$$

Denote  $\lfloor \cdot \rfloor$  to be the integer function, we have the integral form for  $\hat{Y}_t^\epsilon$  as

$$\begin{aligned} \hat{Y}_t^\epsilon &= G_{t/\epsilon} Y_0 + \frac{1}{\epsilon} \int_0^t G_{(t-s)/\epsilon} g(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon) ds \\ &\quad + \frac{1}{\sqrt{\epsilon}} \int_0^t G_{(t-s)/\epsilon} \sigma_2(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon) dW_s^2, \end{aligned}$$

where  $s(\delta) = \lfloor s/\delta \rfloor \delta$  is the nearest breakpoint preceding  $s$ . Define the process  $\hat{X}_t^\epsilon$  by integral

$$\hat{X}_t^\epsilon = S_t' X_0 + S_t \dot{X}_0 + \int_0^t S_{t-s} f(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon) ds + \int_0^t S_{t-s} \sigma_1(X_{s(\delta)}^\epsilon) dW_s^1$$

for  $t \in [0, T_0]$ , We will establish convergence of the auxiliary processes  $\hat{Y}_t^\epsilon$  to the fast solution process  $Y_t^\epsilon$  and  $\hat{X}_t^\epsilon$  to the slow solution process  $X_t^\epsilon$ , respectively.

**Lemma 5.3** *Suppose that conditions in Lemma 5.1 hold. Then there exists a constant  $C > 0$  such that for any  $t \in [0, T_0]$  it holds*

$$\mathbb{E} \|\hat{Y}_t^\epsilon\|^2 \leq C. \quad (5.5)$$

Proof: For  $t \in [0, T_0]$  we have

$$\begin{aligned} \|\hat{Y}_t^\epsilon\|^2 &= \|Y_0\|^2 + \frac{2}{\epsilon} \int_0^t \langle A \hat{Y}_s^\epsilon, \hat{Y}_s^\epsilon \rangle ds + \frac{2}{\epsilon} \int_0^t (g(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon), \hat{Y}_s^\epsilon)_H ds \\ &\quad + \frac{2}{\epsilon} \int_0^t (\hat{Y}_s^\epsilon, dW_s^2)_H + \frac{1}{\epsilon} \int_0^t \|\sigma_2(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon)\|_{\mathcal{R}_2}^2 ds, \text{ a.s..} \end{aligned}$$

This implies that

$$\frac{d}{dt} \mathbb{E} \|\hat{Y}_t^\epsilon\|^2 = \frac{2}{\epsilon} \mathbb{E} \langle A \hat{Y}_t^\epsilon, \hat{Y}_t^\epsilon \rangle + \frac{2}{\epsilon} \mathbb{E} (g(X_{t(\delta)}^\epsilon, \hat{Y}_t^\epsilon), \hat{Y}_t^\epsilon)_H + \frac{1}{\epsilon} \mathbb{E} \|\sigma_2(X_{t(\delta)}^\epsilon, \hat{Y}_t^\epsilon)\|_{\mathcal{R}_2}^2.$$

Using similar arguments for (4.4), we obtain that

$$\frac{d}{dt} \mathbb{E} \|\hat{Y}_t^\epsilon\|^2 \leq -\frac{1}{\epsilon} C_1 \mathbb{E} \|\hat{Y}_t^\epsilon\|^2 + \frac{1}{\epsilon} C_2 (1 + \mathbb{E} \|X_{t(\delta)}^\epsilon\|^2).$$

Then, due to (5.1) we have

$$\frac{d}{dt}\mathbb{E}\|\hat{Y}_t^\epsilon\|^2 \leq -\frac{1}{\epsilon}C_1\mathbb{E}\|\hat{Y}_t^\epsilon\|^2 + \frac{1}{\epsilon}C_2.$$

The Gronwall's inequality yields that

$$\mathbb{E}\|\hat{Y}_t^\epsilon\|^2 \leq C.$$

■

**Lemma 5.4** *There exists a constant  $C > 0$  such that for any  $t \in [0, T_0]$  it holds*

$$\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 \leq C\delta^2, \quad (5.6)$$

$$\mathbb{E}\|X_t^\epsilon - \hat{X}_t^\epsilon\|_1^2 \leq C\delta^2 \quad (5.7)$$

and

$$\mathbb{E}\|\dot{X}_t^\epsilon - \dot{\hat{X}}_t^\epsilon\|^2 \leq C\delta^2. \quad (5.8)$$

Proof: For  $t \in [0, T_0]$  with  $t \in [k\delta, (k+1)\delta]$  we have

$$\begin{aligned} \mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 &= \mathbb{E}\|Y_{k\delta}^\epsilon - \hat{Y}_{k\delta}^\epsilon\|^2 + \frac{2}{\epsilon} \int_{k\delta}^t \mathbb{E}\langle A(Y_s^\epsilon - \hat{Y}_s^\epsilon), Y_s^\epsilon - \hat{Y}_s^\epsilon \rangle ds \\ &\quad + \frac{2}{\epsilon} \int_{k\delta}^t \mathbb{E}(g(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - g(X_s^\epsilon, Y_s^\epsilon), Y_s^\epsilon - \hat{Y}_s^\epsilon)_H ds \\ &\quad + \frac{1}{\epsilon} \int_{k\delta}^t \mathbb{E}\|\sigma_2(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \sigma_2(X_s^\epsilon, Y_s^\epsilon)\|_{\mathcal{R}_2}^2 ds. \end{aligned}$$

This shows that

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 &= \frac{2}{\epsilon}\mathbb{E}\langle A(Y_t^\epsilon - \hat{Y}_t^\epsilon), Y_t^\epsilon - \hat{Y}_t^\epsilon \rangle \\ &\quad + \frac{2}{\epsilon}\mathbb{E}(g(X_{k\delta}^\epsilon, \hat{Y}_t^\epsilon) - g(X_t^\epsilon, Y_t^\epsilon), Y_t^\epsilon - \hat{Y}_t^\epsilon)_H \\ &\quad + \frac{1}{\epsilon}\mathbb{E}\|\sigma_2(X_{k\delta}^\epsilon, \hat{Y}_t^\epsilon) - \sigma_2(X_t^\epsilon, Y_t^\epsilon)\|_{\mathcal{R}_2}^2, \end{aligned} \quad (5.9)$$

which, with the aid of the Young's inequality in the form of  $|ab| \leq \frac{\rho}{2C}|a|^2 + C_\rho|b|^2$  for  $\rho > 0$ , yields from (A5) that

$$\frac{d}{dt}\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 \leq -\frac{C_1}{\epsilon}\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 + \frac{C_2}{\epsilon}\mathbb{E}\|X_{k\delta}^\epsilon - X_t^\epsilon\|^2.$$

By applying (5.4), we have

$$\frac{d}{dt}\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 \leq -\frac{C_1}{\epsilon}\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 + C_2\frac{\delta^2}{\epsilon}$$



and then, due to Gronwall's inequality, we have

$$\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 \leq e^{-C_1 \frac{(t-k\delta)}{\epsilon}} \mathbb{E}\|Y_{k\delta}^\epsilon - \hat{Y}_{k\delta}^\epsilon\|^2 + C_2(1 - e^{-C_1 \frac{(t-k\delta)}{\epsilon}})\delta^2. \quad (5.10)$$

Taking  $t = (k+1)\delta$  in above inequality, one obtains

$$\mathbb{E}\|Y_{(k+1)\delta}^\epsilon - \hat{Y}_{(k+1)\delta}^\epsilon\|^2 \leq \mathbb{E}\|Y_{k\delta}^\epsilon - \hat{Y}_{k\delta}^\epsilon\|^2 e^{-C_1 \frac{\delta}{\epsilon}} + C_2 \left(1 - e^{-C_1 \frac{\delta}{\epsilon}}\right) \delta^2.$$

Iterating the above inequality recursively from  $k$  to 0, we get

$$\begin{aligned} \mathbb{E}\|Y_{(k+1)\delta}^\epsilon - \hat{Y}_{(k+1)\delta}^\epsilon\|^2 &\leq C \left(1 - e^{-C_1 \frac{\delta}{\epsilon}}\right) \delta^2 \sum_{0 \leq j \leq k} e^{-j C_1 \frac{\delta}{\epsilon}} \\ &\leq C \delta^2. \end{aligned}$$

Then, by using again (5.10), we have

$$\mathbb{E}\|Y_t^\epsilon - \hat{Y}_t^\epsilon\|^2 \leq C \delta^2,$$

and therefore the first assertion follows. As for the second estimate, the inequality (2.1) in Proposition 2.1 and (2.4) in Proposition 2.3 imply

$$\mathbb{E}\|X_t^\epsilon - \hat{X}_t^\epsilon\|_1^2 \leq C \int_0^{T_0} \mathbb{E}\|X_s^\epsilon - X_{s(\delta)}^\epsilon\|^2 ds + C \int_0^{T_0} \mathbb{E}\|Y_s^\epsilon - \hat{Y}_s^\epsilon\|^2 ds.$$

From the inequality (5.4) and the first estimate (5.6), this yields the desired estimation (5.7). To show (5.8), we have by (2.2) in Proposition 2.1 and (2.5) in Proposition 2.3 that

$$\begin{aligned} \mathbb{E}\|\dot{X}_t^\epsilon - \dot{\hat{X}}_t^\epsilon\|^2 &\leq T_0 \int_0^{T_0} \mathbb{E}\|f(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon) - f(X_s^\epsilon, Y_s^\epsilon)\|^2 ds \\ &\quad + C \int_0^{T_0} \|\sigma_1(X_{s(\delta)}^\epsilon) - \sigma_1(X_s^\epsilon)\|_{\mathcal{R}_1}^2 ds \\ &\leq C \int_0^{T_0} \mathbb{E}\left(\|X_{s(\delta)}^\epsilon - X_s^\epsilon\|^2 + \|\hat{Y}_s^\epsilon - Y_s^\epsilon\|^2\right) ds \end{aligned}$$

and then thanks to (5.4) and (5.6), we get

$$\mathbb{E}\|\dot{X}_t^\epsilon - \dot{\hat{X}}_t^\epsilon\|^2 \leq C \delta^2.$$

■

## 6 Main result

In this section we will consider the effective dynamics system

$$\frac{\partial^2 \bar{X}_t(\xi)}{\partial t^2} = \Delta \bar{X}_t(\xi) + \bar{f}(\bar{X}_t(\xi)) + \sigma_1(\bar{X}_t(\xi)) \dot{W}_t^1, \quad (6.1)$$

$$\bar{X}_t(\xi) = 0, (\xi, t) \in \partial D \times [0, \infty), \quad (6.2)$$

$$\bar{X}_0(\xi) = X_0(\xi), \frac{\partial \bar{X}_t(\xi)}{\partial t} \Big|_{t=0} = \dot{X}_0(\xi), \xi \in D, \quad (6.3)$$

with

$$\bar{f}(x) = \int_H f(x, y) \mu^x(dy), \quad x \in H,$$

where  $\mu^x$  denotes the unique invariant measure for system (4.1)-(4.3) introduced in Section 4. Moreover, due to [3], the mapping  $\bar{f} : H \mapsto H$  is Lipschitz continuous. Furthermore, by taking (4.6) into account, we have

$$\|f(x, y) - \bar{f}(x)\|^2 \leq C_f (1 + \|x\|^2 + \|y\|^2), \quad x, y \in H. \quad (6.4)$$

The mild form for system (6.1)-(6.3) is

$$\bar{X}_t = S'_t X_0 + S_t \dot{X}_0 + \int_0^t S_{t-s} \bar{f}(\bar{X}_s) ds + \int_0^t S_{t-s} \sigma_1(\bar{X}_s) dW_s^1. \quad (6.5)$$

By arguing as before, the above integral equation admits a unique mild solution in  $L^2(\Omega; C([0, T_0]; H_0^1))$  with its time derivative  $\dot{\bar{X}} \in L^2(\Omega; C([0, T_0]; H))$ . With the above assumptions we have the main result for the slow component, which is proved at the end of this section.

**Theorem 6.1** *Let assumption (A1)-(A5) be fulfilled. In addition, we assume that  $f, \sigma_1, g$  and  $\sigma_2$  are bounded. Then there exists a constant  $C$  such that for any  $t \in [0, T_0]$ ,*

$$\mathbb{E} \left( \|X_t^\epsilon - \bar{X}_t\|_1^2 + \|\dot{X}_t^\epsilon - \dot{\bar{X}}_t\|^2 \right) \leq C\epsilon,$$

here  $\bar{X}_t$  is the solution of the effective system (6.1)-(6.3)

Before proving the Theorems 6.1, we give some lemmas that is needed.

**Lemma 6.1** *Suppose that conditions in Theorem 6.1 hold. Then there exists a constant  $C$  such that for any  $t \in [0, T_0]$ ,*

$$\mathbb{E} \left( \|\hat{X}_t^\epsilon - \bar{X}_t\|_1^2 + \|\hat{\dot{X}}_t^\epsilon - \dot{\bar{X}}_t\|^2 \right) \leq C(\epsilon + \delta^2).$$

Proof: For any  $t \in [0, T_0]$  we write

$$\hat{X}_t^\epsilon - \bar{X}_t = \mathcal{L}_t^\epsilon + \mathcal{M}_t^\epsilon + \mathcal{N}_t^\epsilon + \mathcal{Q}_t^\epsilon + \mathcal{T}_t^\epsilon, \quad (6.6)$$

where

$$\begin{aligned} \mathcal{L}_t^\epsilon &= \int_0^t S_{t-s} (f(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_s^\epsilon)) ds, \\ \mathcal{M}_t^\epsilon &= \int_0^t S_{t-s} (\bar{f}(X_s^\epsilon) - \bar{f}(\hat{X}_s^\epsilon)) ds, \\ \mathcal{N}_t^\epsilon &= \int_0^t S_{t-s} (\bar{f}(\hat{X}_s^\epsilon) - \bar{f}(\bar{X}_s)) ds, \end{aligned}$$

$$\mathcal{Q}_t^\epsilon = \int_0^t S_{t-s}(\sigma_1(X_s^\epsilon) - \sigma_1(\hat{X}_s^\epsilon))dW_s^1$$

and

$$\mathcal{T}_t^\epsilon = \int_0^t S_{t-s}(\sigma_1(\hat{X}_s^\epsilon) - \sigma_1(\bar{X}_s^\epsilon))dW_s^1.$$

By Proposition 2.1, Proposition 2.3 and (5.7), it is easy to show that for any  $t \in [0, T_0]$ ,

$$\begin{aligned} \mathbb{E}\|\mathcal{M}_t^\epsilon\|_1^2 + \mathbb{E}\|\dot{\mathcal{M}}_t^\epsilon\|^2 + \mathbb{E}\|\mathcal{Q}_t^\epsilon\|_1^2 + \mathbb{E}\|\dot{\mathcal{Q}}_t^\epsilon\|^2 &\leq T_0 \int_0^{T_0} \mathbb{E}\|X_s^\epsilon - \hat{X}_s^\epsilon\|^2 ds \\ &\leq C \int_0^{T_0} \mathbb{E}\|X_s^\epsilon - \hat{X}_s^\epsilon\|_1^2 ds \\ &\leq C\delta^2, \end{aligned} \quad (6.7)$$

since  $H$ -norm can be bounded by  $H_0^1$ -norm. By using again the Proposition 2.1 and Proposition 2.3, we have

$$\begin{aligned} \mathbb{E}\|\mathcal{N}_t^\epsilon\|_1^2 + \mathbb{E}\|\dot{\mathcal{N}}_t^\epsilon\|^2 + \mathbb{E}\|\mathcal{T}_t^\epsilon\|_1^2 + \mathbb{E}\|\dot{\mathcal{T}}_t^\epsilon\|^2 &\leq T_0 \int_0^t \mathbb{E}\|\hat{X}_s^\epsilon - \bar{X}_s^\epsilon\|^2 ds \\ &\leq C \int_0^t \mathbb{E}\|\hat{X}_s^\epsilon - \bar{X}_s^\epsilon\|_1^2 ds, \end{aligned} \quad (6.8)$$

Now, due to the inequality (6.7), (6.8) and (6.9) in Lemma 6.2 below, we can get

$$\begin{aligned} &\mathbb{E}\left(\|\hat{X}_t^\epsilon - \bar{X}_t^\epsilon\|_1^2 + \|\dot{\hat{X}}_t^\epsilon - \dot{\bar{X}}_t^\epsilon\|^2\right) \\ &\leq C(\epsilon + \delta^2) + C \int_0^t \mathbb{E}\|\hat{X}_s^\epsilon - \bar{X}_s^\epsilon\|_1^2 ds. \end{aligned}$$

From the Gronwall inequality this yields

$$\mathbb{E}\left(\|\hat{X}_t^\epsilon - \bar{X}_t^\epsilon\|_1^2 + \|\dot{\hat{X}}_t^\epsilon - \dot{\bar{X}}_t^\epsilon\|^2\right) \leq C(\epsilon + \delta^2).$$

Hence the lemma is proved. ■

**Lemma 6.2** Suppose that conditions in Lemma 6.1 hold. Then for any  $T_0 > 0$ , we have

$$\mathbb{E}\left(\|\mathcal{L}_t^\epsilon\|_1^2 + \|\dot{\mathcal{L}}_t^\epsilon\|^2\right) \leq C(\delta^2 + \epsilon), \quad (6.9)$$

where  $C$  is a constant independent of  $(\epsilon, \delta)$ . Proof: For any  $t \in [0, T_0]$ , there exists an  $n_t = \lfloor t/\delta \rfloor$  such that  $t \in [n_t\delta, (n_t + 1)\delta \wedge T_0]$ . Therefore, we have representation in the form

$$\mathcal{L}_t^\epsilon = I_1(t, \epsilon) + I_2(t, \epsilon) + I_3(t, \epsilon), \quad (6.10)$$

where

$$I_1(t, \epsilon) = \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \int_{k\delta}^{(k+1)\delta} S_{t-s} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon) \right) ds,$$

$$\begin{aligned}
 I_2(t, \epsilon) &= \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \int_{k\delta}^{(k+1)\delta} S_{t-s} (\bar{f}(X_{k\delta}^\epsilon) - \bar{f}(X_s^\epsilon)) ds \\
 &= \int_0^{\lfloor t/\delta \rfloor \delta} S_{t-s} (\bar{f}(X_{s(\delta)}^\epsilon) - \bar{f}(X_s^\epsilon)) ds, \\
 I_3(t, \epsilon) &= \int_{\lfloor t/\delta \rfloor \delta}^t S_{t-s} (f(X_{\lfloor t/\delta \rfloor \delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_s^\epsilon)) ds.
 \end{aligned}$$

**Step 1:** Let us first deal with  $I_2(t, \epsilon)$ . Due to the Lipschitz continuity of  $\bar{f}$ , we have the inequalities

$$\begin{aligned}
 \|I_2(t, \epsilon)\|_1^2 &\leq \left[ \int_0^{\lfloor t/\delta \rfloor \delta} \|S_{t-s} (\bar{f}(X_{s(\delta)}^\epsilon) - \bar{f}(X_s^\epsilon))\|_1 ds \right]^2 \\
 &\leq T_0 \int_0^{T_0} \|\bar{f}(X_{s(\delta)}^\epsilon) - \bar{f}(X_s^\epsilon)\|^2 ds \\
 &\leq C \int_0^{T_0} \|X_{s(\delta)}^\epsilon - X_s^\epsilon\|^2 ds,
 \end{aligned}$$

so that, by Lemma 5.2,

$$\mathbb{E} \|I_2(t, \epsilon)\|_1^2 \leq C_{T_0} \delta^2.$$

**Step 2:** We proceed next to the estimation of  $I_3(t, \epsilon)$ . As the mappings  $f : H \times H \rightarrow H$  and  $\bar{f} : H \rightarrow H$  satisfy sublinear growth condition, due to the Hölder inequality we obtain

$$\begin{aligned}
 \|I_3(t, \epsilon)\|_1^2 &\leq \delta \int_{\lfloor t/\delta \rfloor \delta}^t \|S_{t-s} (f(X_{\lfloor t/\delta \rfloor \delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_s^\epsilon))\|_1^2 ds \\
 &\leq \delta C \int_{\lfloor t/\delta \rfloor \delta}^t (1 + \|X_s^\epsilon\|^2 + \|X_{\lfloor t/\delta \rfloor \delta}^\epsilon\|^2 + \|\hat{Y}_s^\epsilon\|^2) ds.
 \end{aligned}$$

This, together with the previous estimate (5.1) and (5.5), allows us to easily get

$$\mathbb{E} \|I_3(t, \epsilon)\|_1^2 \leq C \delta^2.$$

**Step 3:** Let us now deal with  $I_1(t, \epsilon)$ .

Concerning  $I_1(t, \epsilon)$ , we have by the series representation of the Green's function that

$$I_1(t, \epsilon) = \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \mathfrak{i}(t, k, \epsilon),$$

where

$$\begin{aligned}
 &\mathfrak{i}(t, k, \epsilon) \\
 &= \int_{k\delta}^{(k+1)\delta} \sum_{i=1}^{\infty} \frac{\sin\{\sqrt{\alpha_i}(t-s)\}}{\sqrt{\alpha_i}} (f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon), e_i)_H \cdot e_i ds
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^{\infty} \frac{\sin\{\sqrt{\alpha_i}t\}}{\sqrt{\alpha_i}} \cdot e_i \cdot \int_{k\delta}^{(k+1)\delta} \cos\{\sqrt{\alpha_i}s\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \\
 &\quad - \sum_{i=1}^{\infty} \frac{\cos\{\sqrt{\alpha_i}t\}}{\sqrt{\alpha_i}} \cdot e_i \cdot \int_{k\delta}^{(k+1)\delta} \sin\{\sqrt{\alpha_i}s\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds
 \end{aligned}$$

for  $k = 0, 1, \dots, \lfloor t/\delta \rfloor$ , and Clearly,

$$\begin{aligned}
 \|I_1(t, \epsilon)\|_1^2 &= \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \|\mathfrak{I}(t, k, \epsilon)\|_1^2 + 2 \sum_{0 \leq i < j \leq \lfloor t/\delta \rfloor - 1} \left\langle \mathfrak{I}(t, i, \epsilon), \mathfrak{I}(t, j, \epsilon) \right\rangle_1 \\
 &:= \mathcal{A}_1(t, \epsilon) + 2\mathcal{A}_2(t, \epsilon).
 \end{aligned} \tag{6.11}$$

Note that

$$\begin{aligned}
 \|\mathfrak{I}(t, k, \epsilon)\|_1^2 &\leq 2 \sum_{i=1}^{\infty} \left[ \int_{k\delta}^{(k+1)\delta} \cos\{\sqrt{\alpha_i}s\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2 \\
 &\quad + 2 \sum_{i=1}^{\infty} \left[ \int_{k\delta}^{(k+1)\delta} \sin\{\sqrt{\alpha_i}s\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2 \\
 &= 2 \sum_{i=1}^{\infty} \left[ \int_0^\delta \cos\{\sqrt{\alpha_i}(s + k\delta)\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_{s+k\delta}^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2 \\
 &\quad + 2 \sum_{i=1}^{\infty} \left[ \int_0^\delta \sin\{\sqrt{\alpha_i}(s + k\delta)\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_{s+k\delta}^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2.
 \end{aligned}$$

This means

$$\begin{aligned}
 &\mathbb{E}\mathcal{A}_1(t, \epsilon) \\
 &\leq \sum_{k=0}^{\lfloor T_0/\delta \rfloor - 1} \mathbb{E}\|\mathfrak{I}(t, k, \epsilon)\|_1^2 \\
 &\leq C \sum_{k=0}^{\lfloor T_0/\delta \rfloor - 1} \sum_{i=1}^{\infty} \mathbb{E} \left[ \int_0^\delta \cos\{\sqrt{\alpha_i}(s + k\delta)\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_{s+k\delta}^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2 \\
 &\quad + C \sum_{k=0}^{\lfloor T_0/\delta \rfloor - 1} \sum_{i=1}^{\infty} \mathbb{E} \left[ \int_0^\delta \sin\{\sqrt{\alpha_i}(s + k\delta)\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_{s+k\delta}^\epsilon) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2.
 \end{aligned}$$

Note that, by the construction of  $\hat{Y}_t^\epsilon$  and a time shift transformation, we have for any fixed  $k$  and  $s \in [0, \delta)$  the equalities

$$\begin{aligned}
 \hat{Y}_{s+k\delta}^\epsilon &= G_{s/\epsilon} Y_{k\delta}^\epsilon + \frac{1}{\epsilon} \int_{k\delta}^{k\delta+s} G_{(k\delta+s-r)/\epsilon} g(X_{k\delta}^\epsilon, \hat{Y}_r^\epsilon) dr \\
 &\quad + \frac{\sigma_2}{\sqrt{\epsilon}} \int_{k\delta}^{k\delta+s} G_{(k\delta+s-r)/\epsilon} \sigma_2(X_{k\delta}^\epsilon, \hat{Y}_r^\epsilon) dW_r^2 \\
 &= G_{s/\epsilon} Y_{k\delta}^\epsilon + \frac{1}{\epsilon} \int_0^s G_{(s-r)/\epsilon} g(X_{k\delta}^\epsilon, \hat{Y}_{r+k\delta}^\epsilon) dr
 \end{aligned}$$

$$+\frac{\sigma_2}{\sqrt{\epsilon}} \int_0^s G_{(s-r)/\epsilon} \sigma_2(X_{k\delta}^\epsilon, \hat{Y}_{r+k\delta}^\epsilon) dW_r^{*2}, \quad (6.12)$$

where  $W_t^{*2}$  is the shift version of  $W_t^2$  and hence they have the same distribution. Let  $\bar{W}_t$  be a Wiener process defined on the same stochastic basis and independent of  $W_t^1$  and  $W_t^2$ . Construct a process  $Y^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}$  by means of

$$\begin{aligned} Y_{s/\epsilon}^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon} &= G_{s/\epsilon} Y_{k\delta}^\epsilon + \int_0^{s/\epsilon} G_{(s/\epsilon-r)} g(X_{k\delta}^\epsilon, Y_r^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}) dr \\ &\quad + \sigma_2 \int_0^{s/\epsilon} G_{(s/\epsilon-r)} \sigma_2(X_{k\delta}^\epsilon, Y_r^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}) d\bar{W}_r \\ &= G_{s/\epsilon} Y_{k\delta}^\epsilon + \frac{1}{\epsilon} \int_0^s G_{(s-r)/\epsilon} g(X_{k\delta}^\epsilon, Y_{r/\epsilon}^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}) dr \\ &\quad + \frac{\sigma_2}{\sqrt{\epsilon}} \int_0^s G_{(s-r)/\epsilon} \sigma_2(X_{k\delta}^\epsilon, Y_{r/\epsilon}^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}) d\bar{W}_r^\epsilon, \end{aligned} \quad (6.13)$$

where  $\bar{W}_t^\epsilon$  is the scaled version of  $\bar{W}_t$ . By comparison, (6.12) and (6.13) yield

$$(X_{k\delta}^\epsilon, \hat{Y}_{s+k\delta}^\epsilon) \sim (X_{k\delta}^\epsilon, Y_{s/\epsilon}^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}), \quad s \in [0, \delta), \quad (6.14)$$

where  $\sim$  denotes coincidence in distribution sense. In view of (6.14) we have

$$\begin{aligned} \mathbb{E} \mathcal{A}_1(t, \epsilon) &\leq C \sum_{k=0}^{\lfloor T_0/\delta \rfloor - 1} (\mathcal{J}_k^\epsilon + \tilde{\mathcal{J}}_k^\epsilon) \\ &\leq \frac{C}{\delta} \max_{0 \leq k \leq \lfloor T_0/\delta \rfloor} (\mathcal{J}_k^\epsilon + \tilde{\mathcal{J}}_k^\epsilon), \end{aligned}$$

here

$$\begin{aligned} \mathcal{J}_k^\epsilon &:= \sum_{i=1}^{\infty} \mathbb{E} \left[ \int_0^\delta \sin\{\sqrt{\alpha_i}(s+k\delta)\} \left( f(X_{k\delta}^\epsilon, Y_{s/\epsilon}^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2, \\ \tilde{\mathcal{J}}_k^\epsilon &:= \sum_{i=1}^{\infty} \mathbb{E} \left[ \int_0^\delta \cos\{\sqrt{\alpha_i}(s+k\delta)\} \left( f(X_{k\delta}^\epsilon, Y_{s/\epsilon}^{X_{k\delta}^\epsilon, Y_{k\delta}^\epsilon}) - \bar{f}(X_{k\delta}^\epsilon, e_i) \right)_H ds \right]^2 \end{aligned}$$

for  $k = 0, 1, \dots, \lfloor T_0/\delta \rfloor - 1$ . In order to prove Lemma 6.2, we shall need the following lemma, whose proof can be founded in [12, Subsection 6.1]

**Lemma 6.3** Suppose that conditions in Lemma 6.1 hold, then there exists a constant  $C > 0$  such that

$$\mathcal{J}_k^\epsilon \leq C\delta\epsilon \quad (6.15)$$

and

$$\tilde{\mathcal{J}}_k^\epsilon \leq C\delta\epsilon \quad (6.16)$$

for  $k = 0, 1, \dots, \lfloor T_0/\delta \rfloor - 1$ .

Thanks to (6.15) and (6.16), we have

$$\mathbb{E}\mathcal{A}_1(t, \epsilon) \leq C\epsilon. \quad (6.17)$$

Next, by using the idea introduced in [2] and [20], let us estimate  $\mathbb{E}\mathcal{A}_2(t, \epsilon)$ . We introduce Markov processes that generalize  $\hat{Y}_t^\epsilon$ . For  $k = 0, 1, \dots, \lfloor T_0/\delta \rfloor$ , we denote by  $\{Z_t^{k, \epsilon}\}_{t \geq k\delta}$  the solution of the problem

$$\begin{aligned} dZ_t^{k, \epsilon} &= \frac{1}{\epsilon} (AZ_t^{k, \epsilon} + g(X_{k\delta}^\epsilon, Z_t^{k, \epsilon})) dt \\ &\quad + \frac{1}{\sqrt{\epsilon}} \sigma_2(X_{k\delta}^\epsilon, Z_t^{k, \epsilon}) dW_t^2, \quad t \geq k\delta, \\ Z_{k\delta}^{k, \epsilon} &= \hat{Y}_{k\delta}^\epsilon. \end{aligned}$$

It is immediate to check that if  $\tau \in [k\delta, (k+1)\delta)$ , we have

$$\hat{Y}_\tau^\epsilon = Z_\tau^{k, \epsilon}.$$

Also, the continuity implies that

$$Z_{(k+1)\delta}^{k, \epsilon} = \hat{Y}_{(k+1)\delta}^\epsilon = Z_{(k+1)\delta}^{k+1, \epsilon}.$$

For  $i\delta \leq s \leq (i+1)\delta \leq j\delta \leq \tau \leq (j+1)\delta$ , we have

$$\begin{aligned} &\left\langle \mathfrak{I}(t, i, \epsilon), \mathfrak{I}(t, j, \epsilon) \right\rangle_1 \\ &= \left\langle \int_{i\delta}^{(i+1)\delta} S_{t-s} [f(X_{i\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{i\delta}^\epsilon)] ds, \right. \\ &\quad \left. \int_{j\delta}^{(j+1)\delta} S_{t-\tau} [f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon)] d\tau \right\rangle_1 \\ &= \sum_{n=1}^{\infty} \left\{ \sqrt{\alpha_n} \int_{i\delta}^{(i+1)\delta} \left( S_{t-s} [f(X_{i\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{i\delta}^\epsilon)], e_n \right)_H ds \right. \\ &\quad \left. \times \sqrt{\alpha_n} \int_{j\delta}^{(j+1)\delta} \left( S_{t-\tau} [f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon)], e_n \right)_H d\tau \right\} \\ &= \int_{i\delta}^{(i+1)\delta} \int_{j\delta}^{(j+1)\delta} \sum_{n=1}^{\infty} \left[ \alpha_n \left( S_{t-s} [f(X_{i\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{i\delta}^\epsilon)], e_n \right)_H \right. \\ &\quad \left. \times \left( S_{t-\tau} [f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon)], e_n \right)_H \right] ds d\tau. \end{aligned}$$

From this, one sees that

$$\begin{aligned} &\mathbb{E} \left\langle \mathfrak{I}(t, i, \epsilon), \mathfrak{I}(t, j, \epsilon) \right\rangle_1 \\ &= \int_{i\delta}^{(i+1)\delta} \int_{j\delta}^{(j+1)\delta} \mathbb{E} \sum_{n=1}^{\infty} \left\{ \alpha_n \left( S_{t-s} [f(X_{i\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{i\delta}^\epsilon)], e_n \right)_H \right. \end{aligned}$$

$$\begin{aligned}
 & \times \mathbb{E} \left[ \left( S_{t-\tau} [f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon)], e_n \right)_H \middle| \mathcal{F}_{(i+1)\delta} \right] \Big\} ds d\tau \\
 &= \int_{i\delta}^{(i+1)\delta} \int_{j\delta}^{(j+1)\delta} \mathbb{E} \sum_{n=1}^{\infty} \left\{ \alpha_n \left( S_{t-s} [f(X_{i\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{i\delta}^\epsilon)], e_n \right)_H \right. \\
 & \quad \times \left( \mathbb{E} \left[ S_{t-\tau} [f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon)] \middle| \mathcal{F}_{(i+1)\delta} \right], e_n \right)_H \Big\} ds d\tau \\
 &= \int_{i\delta}^{(i+1)\delta} \int_{j\delta}^{(j+1)\delta} \mathbb{E} \left\langle S_{t-s} [f(X_{i\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{i\delta}^\epsilon)], \right. \\
 & \quad \left. \mathbb{E} \left[ S_{t-\tau} [f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon)] \middle| \mathcal{F}_{(i+1)\delta} \right] \right\rangle_1 ds d\tau \\
 &= \int_{i\delta}^{(i+1)\delta} \int_{j\delta}^{(j+1)\delta} \mathbb{E} \left\langle S_{t-s} [f(X_{i\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{i\delta}^\epsilon)], \right. \\
 & \quad \left. S_{t-\tau} \mathbb{E} \left[ f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon) \middle| \mathcal{F}_{(i+1)\delta} \right] \right\rangle_1 ds d\tau.
 \end{aligned}$$

Hence, by boundness of  $f$  we have

$$\begin{aligned}
 & \mathbb{E} \left\langle \mathfrak{I}(t, i, \epsilon), \mathfrak{I}(t, j, \epsilon) \right\rangle_1 \\
 & \leq C \int_{i\delta}^{(i+1)\delta} \int_{j\delta}^{(j+1)\delta} \mathbb{E} \left\| S_{t-\tau} \mathbb{E} \left[ f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon) \middle| \mathcal{F}_{(i+1)\delta} \right] \right\|_1 ds d\tau \\
 & \leq C \int_{i\delta}^{(i+1)\delta} \int_{j\delta}^{(j+1)\delta} \mathbb{E} \left\| \mathbb{E} \left[ f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon) \middle| \mathcal{F}_{(i+1)\delta} \right] \right\| ds d\tau. \\
 & \leq C\delta \int_{j\delta}^{(j+1)\delta} \mathbb{E} \left\| \mathbb{E} \left[ f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon) \middle| \mathcal{F}_{(i+1)\delta} \right] \right\| d\tau.
 \end{aligned}$$

Since  $\hat{Y}_\tau^\epsilon = Z_\tau^{j,\epsilon}$ , we can study the integrand above by considering the decomposition

$$\begin{aligned}
 & \mathbb{E} \left\| \mathbb{E} \left[ f(X_{j\delta}^\epsilon, \hat{Y}_\tau^\epsilon) - \bar{f}(X_{j\delta}^\epsilon) \middle| \mathcal{F}_{(i+1)\delta} \right] \right\| \\
 & \leq \mathbb{E} \left\| \mathbb{E} \left[ f(X_{(i+1)\delta}^\epsilon, Z_\tau^{i+1,\epsilon}) - \bar{f}(X_{(i+1)\delta}^\epsilon) \middle| \mathcal{F}_{(i+1)\delta} \right] \right\| \\
 & \quad + \mathbb{E} \left\| \mathbb{E} \left[ \left( f(X_{j\delta}^\epsilon, Z_\tau^{j,\epsilon}) - \bar{f}(X_{j\delta}^\epsilon) \right) - \left( f(X_{(i+1)\delta}^\epsilon, Z_\tau^{i+1,\epsilon}) \right. \right. \right. \\
 & \quad \left. \left. \left. - \bar{f}(X_{(i+1)\delta}^\epsilon) \right) \middle| \mathcal{F}_{(i+1)\delta} \right] \right\| \\
 & := B_1 + B_2.
 \end{aligned}$$

In what follows, we denote by  $\{Z_t^{\epsilon,x,y}\}_{t \geq 0}$  the solution of equation

$$dZ_t^\epsilon = \frac{1}{\epsilon} (AZ_t^\epsilon + g(x, Z_t^\epsilon)) dt + \frac{1}{\sqrt{\epsilon}} \sigma_2(x, Z_t^\epsilon) d\tilde{W}_t, \quad Z_0^\epsilon = y,$$

where the  $\mathcal{R}_2$ -Wiener processes  $\tilde{W}_t$  is independent of  $W_t^1$  and  $W_t^2$ . It is clear that, for any  $k = 0, 1, \dots, \lfloor T_0/\delta \rfloor$ , the distribution of the process

$$Z_t^{k,\epsilon}, \quad t \geq k\delta,$$



coincides with the distribution of the process

$$Z_{t-k\delta}^{\epsilon, X_{k\delta}^{\epsilon}, \hat{Y}_{k\delta}^{\epsilon}}, \quad t \geq k\delta.$$

Hence, thanks to Markov property, we obtain

$$\begin{aligned} B_1 &= \mathbb{E} \left\| \mathbb{E} \left[ f(x, Z_{\tau-(i+1)\delta}^{\epsilon, x, y}) - \bar{f}(x) \right] \Big|_{x=X_{(i+1)\delta}^{\epsilon}, y=\hat{Y}_{(i+1)\delta}^{\epsilon}} \right\| \\ &= \mathbb{E} \left\| \mathbb{E} \left[ f(x, Y_{\frac{\tau-(i+1)\delta}{\epsilon}}^{x, y}) - \bar{f}(x) \right] \Big|_{x=X_{(i+1)\delta}^{\epsilon}, y=\hat{Y}_{(i+1)\delta}^{\epsilon}} \right\|, \end{aligned}$$

where  $Y^{x, y}$  denotes the solution of problem (4.1)-(4.3) with slow component  $x$  and initial datum  $y$ . Then by (4.7), we have

$$B_1 \leq Ce^{-c\frac{\tau-(i+1)\delta}{2\epsilon}} (1 + \mathbb{E}\|X_{(i+1)\delta}^{\epsilon}\| + \mathbb{E}\|\hat{Y}_{(i+1)\delta}^{\epsilon}\|).$$

Thus, by Lemma 5.1 and Lemma 5.3 we have

$$B_1 \leq Ce^{-c\frac{\tau-(i+1)\delta}{2\epsilon}}.$$

Next, let us estimate  $B_2$ . Thanks to the tower property of conditional expectation, we obtain

$$\begin{aligned} B_2 &= \mathbb{E} \left\| \mathbb{E} \left( \mathbb{E} \left[ \sum_{m=i+1}^{j-1} [f(X_{(m+1)\delta}^{\epsilon}, Z_{\tau}^{m+1, \epsilon}) - \bar{f}(X_{(m+1)\delta}^{\epsilon})] \Big| \mathcal{F}_{(m+1)\delta} \right] \Big| \mathcal{F}_{(i+1)\delta} \right) \right. \\ &\quad \left. - \mathbb{E} \left( \mathbb{E} \left[ \sum_{m=i+1}^{j-1} [f(X_{m\delta}^{\epsilon}, Z_{\tau}^{m, \epsilon}) - \bar{f}(X_{m\delta}^{\epsilon})] \Big| \mathcal{F}_{(m+1)\delta} \right] \Big| \mathcal{F}_{(i+1)\delta} \right) \right\| \\ &\leq \sum_{m=i+1}^{j-1} \mathbb{E} \left\| \mathbb{E} \left[ f(X_{(m+1)\delta}^{\epsilon}, Z_{\tau}^{m+1, \epsilon}) - \bar{f}(X_{(m+1)\delta}^{\epsilon}) \Big| \mathcal{F}_{(m+1)\delta} \right] \right. \\ &\quad \left. - \mathbb{E} \left[ f(X_{m\delta}^{\epsilon}, Z_{\tau}^{m, \epsilon}) - \bar{f}(X_{m\delta}^{\epsilon}) \Big| \mathcal{F}_{(m+1)\delta} \right] \right\|. \end{aligned}$$

Due to the Markov property, we have

$$\begin{aligned} &\mathbb{E} \left[ f(X_{(m+1)\delta}^{\epsilon}, Z_{\tau}^{m+1, \epsilon}) - \bar{f}(X_{(m+1)\delta}^{\epsilon}) \Big| \mathcal{F}_{(m+1)\delta} \right] \\ &= \mathbb{E} \left[ f(x, Z_{\tau-(m+1)\delta}^{\epsilon, x, y}) - \bar{f}(x) \right] \Big|_{x=X_{(m+1)\delta}^{\epsilon}, y=\hat{Y}_{(m+1)\delta}^{\epsilon}} \\ &= \mathbb{E} \left[ f(x, Y_{\frac{\tau-(m+1)\delta}{\epsilon}}^{\epsilon, x, y}) - \bar{f}(x) \right] \Big|_{x=X_{(m+1)\delta}^{\epsilon}, y=\hat{Y}_{(m+1)\delta}^{\epsilon}}. \end{aligned}$$

Similarly, one has

$$\begin{aligned} &\mathbb{E} \left[ f(X_{m\delta}^{\epsilon}, Z_{\tau}^{m, \epsilon}) - \bar{f}(X_{m\delta}^{\epsilon}) \Big| \mathcal{F}_{(m+1)\delta} \right] \\ &= \mathbb{E} \left[ f(x, Z_{\tau-(m+1)\delta}^{\epsilon, x, y}) - \bar{f}(x) \right] \Big|_{x=X_{m\delta}^{\epsilon}, y=\hat{Y}_{(m+1)\delta}^{\epsilon}} \\ &= \mathbb{E} \left[ f(x, Y_{\frac{\tau-(m+1)\delta}{\epsilon}}^{\epsilon, x, y}) - \bar{f}(x) \right] \Big|_{x=X_{m\delta}^{\epsilon}, y=\hat{Y}_{(m+1)\delta}^{\epsilon}}. \end{aligned}$$

Let us define

$$\tilde{f}(x, y, t) := \mathbb{E}f(x, Y_t^{x,y}) - \bar{f}(x), \quad x, y \in H. \quad (6.18)$$

Hence, thanks to Lemma 7.1 presented in the final section, we have

$$\begin{aligned} B_2 &\leq \sum_{m=i+1}^{j-1} \mathbb{E} \left\| \tilde{f}(X_{m\delta}^\epsilon, Y_{(m+1)\delta}^\epsilon, \frac{\tau - (m+1)\delta}{\epsilon}) \right. \\ &\quad \left. - \tilde{f}(X_{(m+1)\delta}^\epsilon, Y_{(m+1)\delta}^\epsilon, \frac{\tau - (m+1)\delta}{\epsilon}) \right\| \\ &\leq C \sum_{m=i+1}^{j-1} e^{-c\frac{\tau - (m+1)\delta}{\epsilon}} (\mathbb{E} \|X_{(m+1)\delta}^\epsilon - X_{m\delta}^\epsilon\|) (1 + \mathbb{E} \|\hat{Y}_{(m+1)\delta}^\epsilon\|) \\ &\leq C\delta \sum_{m=i+1}^{j-1} e^{-c\frac{\tau - (m+1)\delta}{\epsilon}} \\ &\leq C\delta \frac{e^{-c\frac{\tau - j\delta}{\epsilon}}}{1 - e^{-c\frac{\delta}{\epsilon}}}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \mathbb{E}A_2(t, \epsilon) &\leq C \sum_{0 \leq i < j \leq \lfloor t/\delta \rfloor - 1} \delta \int_{j\delta}^{(j+1)\delta} e^{-c\frac{\tau - (i+1)\delta}{\epsilon}} d\tau \\ &\quad + C \sum_{0 \leq i < j \leq \lfloor t/\delta \rfloor - 1} \delta \int_{j\delta}^{(j+1)\delta} \delta \frac{e^{-c\frac{\tau - j\delta}{\epsilon}}}{1 - e^{-c\frac{\delta}{\epsilon}}} d\tau \\ &\leq C \sum_{0 \leq i < j \leq \lfloor t/\delta \rfloor - 1} \left[ e^{-\frac{c}{\epsilon}(j-i-1)\delta} (1 - e^{-\frac{c}{\epsilon}\delta}) \delta \epsilon + \delta^2 \epsilon \right] \\ &\leq C\epsilon. \end{aligned}$$

This fact, together with estimate (6.17) and equality (6.11), show

$$\mathbb{E} \|I(t, \epsilon)\|_1^2 \leq C\epsilon. \quad (6.19)$$

**Step 4:** Estimate of  $\mathbb{E} \|\mathcal{L}_t^\epsilon\|_1^2$ .

It is now easy to gather all previous estimates for terms in (6.10) and deduce

$$\mathbb{E} \|\mathcal{L}_t^\epsilon\|_1^2 \leq C(\delta^2 + \epsilon).$$

**Step 5:** Estimate of  $\mathbb{E} \|\dot{\mathcal{L}}_t^\epsilon\|^2$ .

It is also easy to see that

$$\begin{aligned} \dot{\mathcal{L}}_t^\epsilon &= \sum_{i=1}^{\infty} e_i \cdot \int_0^t \cos\{\sqrt{\alpha_i}(t-s)\} \left( f(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_s^\epsilon), e_i \right)_H ds \\ &= \int_0^t S'_{t-s} \left( f(X_{s(\delta)}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_s^\epsilon) \right) ds, \end{aligned}$$

and so that we have the decomposition

$$\dot{\mathcal{L}}_t^\epsilon = \tilde{I}_1(t, \epsilon) + \tilde{I}_2(t, \epsilon) + \tilde{I}_3(t, \epsilon), \quad (6.20)$$

where

$$\begin{aligned} \tilde{I}_1(t, \epsilon) &= \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \int_{k\delta}^{(k+1)\delta} S'_{t-s} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon) \right) ds, \\ \tilde{I}_2(t, \epsilon) &= \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \int_{k\delta}^{(k+1)\delta} S'_{t-s} \left( \bar{f}(X_{k\delta}^\epsilon) - \bar{f}(X_s^\epsilon) \right) ds \\ &= \int_0^{\lfloor t/\delta \rfloor \delta} S'_{t-s} \left( \bar{f}(X_{s(\delta)}^\epsilon) - \bar{f}(X_s^\epsilon) \right) ds, \\ \tilde{I}_3(t, \epsilon) &= \int_{\lfloor t/\delta \rfloor \delta}^t S'_{t-s} \left( f(X_{\lfloor t/\delta \rfloor \delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_s^\epsilon) \right) ds. \end{aligned}$$

For  $\tilde{I}_2(t, \epsilon)$ , we have by the Hölder inequality and the Lipschitz continuity of  $\bar{f}$

$$\begin{aligned} \|\tilde{I}_2(t, \epsilon)\|^2 &\leq \left[ \int_0^{\lfloor t/\delta \rfloor \delta} \left\| S'_{t-s} \left( \bar{f}(X_{s(\delta)}^\epsilon) - \bar{f}(X_s^\epsilon) \right) \right\| ds \right]^2 \\ &\leq T_0 \int_0^{T_0} \left\| \bar{f}(X_{s(\delta)}^\epsilon) - \bar{f}(X_s^\epsilon) \right\|^2 ds \\ &\leq C \int_0^{T_0} \left\| X_{s(\delta)}^\epsilon - X_s^\epsilon \right\|^2 ds. \end{aligned}$$

By virtue of (5.4) in Lemma 5.2 we obtain

$$\mathbb{E} \|\tilde{I}_2(t, \epsilon)\|^2 \leq C\delta^2. \quad (6.21)$$

For  $\tilde{I}_3(t, \epsilon)$ , we have by the Hölder inequality

$$\begin{aligned} \|\tilde{I}_3(t, \epsilon)\|^2 &\leq \delta \int_{\lfloor t/\delta \rfloor \delta}^t \left\| S'_{t-s} \left( f(X_{\lfloor t/\delta \rfloor \delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_s^\epsilon) \right) \right\|^2 ds \\ &\leq \delta C \int_{\lfloor t/\delta \rfloor \delta}^t \left( 1 + \|X_s^\epsilon\|^2 + \|X_{\lfloor t/\delta \rfloor \delta}^\epsilon\|^2 + \|\hat{Y}_s^\epsilon\|^2 \right) ds. \end{aligned}$$

Therefore, we easily have

$$\mathbb{E} \|\tilde{I}_3(t, \epsilon)\|^2 \leq C\delta^2. \quad (6.22)$$

For  $\tilde{I}_1(t, \epsilon)$ , we have by the series representation of the Green's function that

$$\tilde{I}_1(t, \epsilon) = \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \tilde{\mathfrak{I}}(t, k, \epsilon),$$

where

$$\begin{aligned}
 & \tilde{\mathbf{i}}(t, k, \epsilon) \\
 &:= \int_{k\delta}^{(k+1)\delta} \sum_{i=1}^{\infty} \cos\{\sqrt{\alpha_i}(t-s)\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon), e_i \right)_H \cdot e_i ds \\
 &= \sum_{i=1}^{\infty} \cos\{\sqrt{\alpha_i}t\} \cdot e_i \cdot \int_{k\delta}^{(k+1)\delta} \cos\{\sqrt{\alpha_i}s\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon), e_i \right)_H ds \\
 &+ \sum_{i=1}^{\infty} \sin\{\sqrt{\alpha_i}t\} \cdot e_i \cdot \int_{k\delta}^{(k+1)\delta} \sin\{\sqrt{\alpha_i}s\} \left( f(X_{k\delta}^\epsilon, \hat{Y}_s^\epsilon) - \bar{f}(X_{k\delta}^\epsilon), e_i \right)_H ds
 \end{aligned}$$

for  $k = 0, 1, \dots, \lfloor t/\delta \rfloor - 1$ . It is clear that

$$\begin{aligned}
 \|\tilde{I}_1(t, \epsilon)\|^2 &= \sum_{k=0}^{\lfloor t/\delta \rfloor - 1} \|\tilde{\mathbf{i}}(t, k, \epsilon)\|^2 + 2 \sum_{0 \leq i < j \leq \lfloor t/\delta \rfloor - 1} \left( \tilde{\mathbf{i}}(t, i, \epsilon), \tilde{\mathbf{i}}(t, j, \epsilon) \right)_H \\
 &:= \tilde{\mathcal{A}}_1(t, \epsilon) + 2\tilde{\mathcal{A}}_2(t, \epsilon).
 \end{aligned}$$

Repeating the same argument as in Step 3, we can conclude that

$$\mathbb{E}\tilde{\mathcal{A}}_1(t, \epsilon) \leq C\epsilon$$

and

$$\mathbb{E}\tilde{\mathcal{A}}_2(t, \epsilon) \leq C\epsilon,$$

which means

$$\mathbb{E}\|\tilde{I}_1(t, \epsilon)\|^2 \leq C\epsilon. \quad (6.23)$$

Now, in view of estimates (6.21), (6.22) and (6.23), from (6.20) we obtain

$$\mathbb{E}\|\dot{\mathcal{L}}_t^\epsilon\|^2 \leq C(\epsilon + \delta^2).$$

**Step 6: Conclusion.** By making use of the results in Step 4 and Step 5, we get the desired inequality (6.9). ■

## 6.1 Proof of Theorem 6.1

*Proof:* According to (5.7), (5.8) and to Lemma 6.1, we have

$$\mathbb{E} \left( \|X_t^\epsilon - \bar{X}_t\|_1^2 + \|\dot{X}_t^\epsilon - \dot{\bar{X}}_t\|^2 \right) \leq C(\epsilon + \delta^2).$$

Taking  $\delta = \sqrt{\epsilon}$ , we obtain

$$\mathbb{E} \left( \|X_t^\epsilon - \bar{X}_t\|_1^2 + \|\dot{X}_t^\epsilon - \dot{\bar{X}}_t\|^2 \right) \leq C\epsilon,$$

which completes the proof. ■

## 7 Auxiliary Lemma

In this section, we state and prove some technical lemmas used in the former section.

**Lemma 7.1** *Function  $\tilde{f}$  defined by (6.18) is Lipschitz continuous with respect to  $x$ . In addition, there exist  $c, C > 0$ , such that for any  $x_1, x_2, y \in H$  and  $t > 0$  we have*

$$\|\tilde{f}(x_1, y, t) - \tilde{f}(x_2, y, t)\| \leq C(1 + \|y\|)\|x_1 - x_2\|e^{-ct}.$$

Proof: We shall follow the approach of [2, Proposition C.2]. For any  $t_0 > 0$ , we set

$$\tilde{F}_{t_0}(x, y, t) = F(x, y, t) - F(x, y, t + t_0),$$

where

$$F(x, y, t) := \mathbb{E}f(x, Y_t^{x,y}).$$

Thanks to Markov property we then write that

$$\begin{aligned} \tilde{F}_{t_0}(x, y, t) &= F(x, y, t) - \mathbb{E}f(x, Y_{t+t_0}^{x,y}) \\ &= F(x, y, t) - \mathbb{E}F(x, Y_{t_0}^{x,y}, t) \end{aligned}$$

In view of the assumption (A1),  $F$  is Gâteaux-differentiable with respect to  $x$  at  $(x, y, t)$ . Therefore, we have for any  $h \in H$  that

$$\begin{aligned} D_x \tilde{F}_{t_0}(x, y, t) \cdot h &= D_x F(x, y, t) \cdot h - \mathbb{E}D_x (F(x, Y_{t_0}^{x,y}, t)) \cdot h \\ &= F'_x(x, y, t) \cdot h - \mathbb{E}F'_x(x, Y_{t_0}^{x,y}, t) \cdot h \\ &\quad - \mathbb{E}F'_y(x, Y_{t_0}^{x,y}, t) \cdot (D_x Y_{t_0}^{x,y} \cdot h), \end{aligned} \tag{7.1}$$

where we use the symbol  $F'_x$  and  $F'_y$  to denote the Gâteaux derivative with respect to  $x$  and  $y$ , respectively. Note that the first derivative  $\zeta_t^{x,y,h} = D_x Y_t^{x,y} \cdot h$ , at the point  $x$  and along the direction  $h \in H$ , is the solution of equation

$$\begin{aligned} d\zeta_t^{x,y,h} &= \left( A\zeta_t^{x,y,h} + g'_x(x, Y_t^{x,y}) \cdot h + g'_y(x, Y_t^{x,y}) \cdot \zeta_t^{x,y,h} \right) dt \\ &\quad + \left( \sigma'_{2,x}(x, Y_t^{x,y}) \cdot h + \sigma'_{2,y}(x, Y_t^{x,y}) \cdot \zeta_t^{x,y,h} \right) dW_t^2 \end{aligned}$$

with initial data  $\zeta_0^{x,y,h} = 0$ . Hence, thanks to (A3), it is immediate to check that for any  $t \geq 0$ ,

$$\mathbb{E}\|\zeta_t^{x,y,h}\| \leq C\|h\|. \tag{7.2}$$

Notice that for any  $y_1, y_2 \in H$ , we have

$$\begin{aligned} \|F(x, y_1, t) - F(x, y_2, t)\| &= \|\mathbb{E}f(x, Y_t^{x, y_1}) - \mathbb{E}f(x, Y_t^{x, y_2})\| \\ &\leq C\mathbb{E}\|Y_t^{x, y_1} - Y_t^{x, y_2}\| \\ &\leq Ce^{-ct}\|y_1 - y_2\|, \end{aligned}$$

which implies

$$\|F'_y(x, y, t) \cdot k\| \leq Ce^{-ct}\|k\|, \quad k \in H. \quad (7.3)$$

Therefore, thanks to (7.2) and (7.3), we can conclude that

$$\|\mathbb{E}[F'_y(x, Y_{t_0}^{x, y}, t) \cdot (D_x Y_{t_0}^{x, y} \cdot h)]\| \leq Ce^{-ct}\|h\|. \quad (7.4)$$

Then, we directly have

$$\begin{aligned} &F'_x(x, y_1, t) \cdot h - F'_x(x, y_2, t) \cdot h \\ &= \mathbb{E}(f'_x(x, Y_t^{x, y_1})) \cdot h - \mathbb{E}(f'_x(x, Y_t^{x, y_2})) \cdot h \\ &\quad + \mathbb{E}\left(f'_y(x, Y_t^{x, y_1}) \cdot \zeta_t^{x, y_1, h} - f'_y(x, Y_t^{x, y_2}) \cdot \zeta_t^{x, y_2, h}\right) \\ &= \mathbb{E}(f'_x(x, Y_t^{x, y_1})) \cdot h - \mathbb{E}(f'_x(x, Y_t^{x, y_2})) \cdot h \\ &\quad + \mathbb{E}\left([f'_y(x, Y_t^{x, y_1}) - f'_y(x, Y_t^{x, y_2})] \cdot \zeta_t^{x, y_1, h}\right) \\ &\quad + \mathbb{E}\left(f'_y(x, Y_t^{x, y_2}) \cdot (\zeta_t^{x, y_1, h} - \zeta_t^{x, y_2, h})\right). \end{aligned} \quad (7.5)$$

First it is easy to show

$$\begin{aligned} &\|\mathbb{E}(f'_x(x, Y_t^{x, y_1})) \cdot h - \mathbb{E}(f'_x(x, Y_t^{x, y_2})) \cdot h\| \\ &\leq \mathbb{E}\|(f'_x(x, Y_t^{x, y_1})) \cdot h - (f'_x(x, Y_t^{x, y_2})) \cdot h\| \\ &\leq C\mathbb{E}\|Y_t^{x, y_1} - Y_t^{x, y_2}\| \cdot \|h\| \\ &\leq Ce^{-ct}\|y_1 - y_2\| \cdot \|h\|. \end{aligned} \quad (7.6)$$

Next, by Assumption (A1) we have

$$\begin{aligned} &\|\mathbb{E}\left([f'_y(x, Y_t^{x, y_1}) - f'_y(x, Y_t^{x, y_2})] \cdot \zeta_t^{x, y_1, h}\right)\| \\ &\leq \mathbb{E}\|[f'_y(x, Y_t^{x, y_1}) - f'_y(x, Y_t^{x, y_2})] \cdot \zeta_t^{x, y_1, h}\| \\ &\leq C\{\mathbb{E}\|\zeta_t^{x, y_1, h}\|^2\}^{\frac{1}{2}} \cdot \{\mathbb{E}\|Y_t^{x, y_1} - Y_t^{x, y_2}\|^2\}^{\frac{1}{2}} \\ &\leq Ce^{-ct}\|h\| \cdot \|y_1 - y_2\|. \end{aligned} \quad (7.7)$$

By making use of Assumption (A1) again, we can show

$$\begin{aligned} &\|\mathbb{E}\left(f'_y(x, Y_t^{x, y_2}) \cdot (\zeta_t^{x, y_1, h} - \zeta_t^{x, y_2, h})\right)\| \\ &\leq \mathbb{E}\left\|f'_y(x, Y_t^{x, y_2}) \cdot (\zeta_t^{x, y_1, h} - \zeta_t^{x, y_2, h})\right\| \end{aligned}$$

$$\begin{aligned} &\leq C\mathbb{E}\|\zeta_t^{x,y_1,h} - \zeta_t^{x,y_2,h}\| \\ &\leq Ce^{-c't}\|y_1 - y_2\| \cdot \|h\|. \end{aligned} \tag{7.8}$$

Collecting together (7.5), (7.6), (7.7) and (7.8), we get

$$\begin{aligned} &\|F'_x(x, y_1, t) \cdot h - F'_x(x, y_2, t) \cdot h\| \\ &= \leq Ce^{-c_0t}\|y_1 - y_2\| \cdot \|h\|, \end{aligned}$$

which means

$$\begin{aligned} &\|F'_x(x, y, t) \cdot h - F'_x(x, Y_{t_0}^{x,y}, t) \cdot h\| \\ &\leq Ce^{-c_0t}(1 + \|y\|) \cdot \|h\| \end{aligned} \tag{7.9}$$

since

$$\mathbb{E}\|Y_{t_0}^{x,y}\| \leq C(1 + \|y\|).$$

Returning to (7.1), by (7.4) and (7.9) we conclude that

$$\|D_x \tilde{F}_{t_0}(x, y, t) \cdot h\| \leq Ce^{-ct}(1 + \|y\|)\|h\|.$$

This yields

$$\|\tilde{F}_{t_0}(x_1, y, t) - \tilde{F}_{t_0}(x_2, y, t)\| \leq Ce^{-ct}(1 + \|y\|)\|x_1 - x_2\|.$$

Letting  $t_0 \rightarrow +\infty$ , we obtain

$$\|\tilde{f}(x_1, y, t) - \tilde{f}(x_2, y, t)\| \leq C(1 + \|y\|)\|x_1 - x_2\|e^{-ct}.$$

■

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## References

- [1] Bogoliubov, N. N., and Mitropolsky Y. A. 1961. *Asymptotic Methods in the Theory of Non-linear Oscillations*. New York: Gordon & Breach Science.
- [2] Bréhier, C. E. 2012. Strong and weak orders in averging for SPDEs. *Stochastic Process. Appl.* 122:2553-2593.

- [3] Cerrai, S. and Freidlin, M. I. 2009. Averaging principle for a class of stochastic reaction-diffusion equations. *Proba. Theory Related Fields* 144:137-177.
- [4] Cerrai, S. 2009. A Khasminkii type averaging principle for stochastic reaction-diffusion equations. *Ann. Appl. Probab.* 19:899-948.
- [5] Cerrai, S. 2011. Averaging principle for systems of reaction-diffusion equations with polynomial nonlinearities perturbed by multiplicative type noise. *SIAM J. Math. Anal.* 43:2482-2518.
- [6] Choi, Y. S. and Miller, C. 2011, Global existence of solutions to a coupled parabolic-hyperbolic system with moving boundary. *Proc. Amer. Math. Soc.* 139:3257-3270.
- [7] Chow, P. L. 1973. Thermoelastic wave propagation in a random medium and some related problems. *Int. J. Eng. Sci.* 11:253-971.
- [8] Chow, P. L. 2007. *Stochastic Partial Differential Equations*. New York: Chapman & Hall/CRC.
- [9] Cardetti, F. and Choi, Y. S. 2009. A parabolic-hyperbolic system modelling a moving cell. *Electron. J. Differential Equations* 95:1-11.
- [10] Freidlin, M. I. and Wentzell, A. D. 1998. *Random Perturbation of Dynamical Systems*, 2nd ed. New York: Springer-Verlag.
- [11] Freidlin, M. I. and Wentzell, A. D. 2006. Long-time behavior of weakly coupled oscillators. *J. Stat. Phys.* 123:1311-1337.
- [12] Fu, H., Wan, L. and Liu, J. 2015. Strong convergence in averaging principle for stochastic hyperbolic-parabolic equations with two time-scales. *Stochastic Process. Appl.* 125:3255-3279.
- [13] Khasminskii, R. Z. 1968. On the principle of averaging the Itô stochastic differential equations. *Kibernetika* 4:260-279. [in Russian]
- [14] Khasminskii, R. Z. and Yin, G. 2005. Limit behavior of two-time-scale diffusions revisited. *J. Differential Equations* 212:85-113.
- [15] Kifer, Y. 2004. *Some recent advance in averaging, Modern dynamical systems and applications*. UK: Cambridge University Press.
- [16] Kifer, Y. 2004. Diffusion approximation for slow motion in fully coupled averaging. *Proba. Theor. Relat. Fields* 129: 157-181.
- [17] Kifer, Y. 2005. Another proof of the averaging principle for fully coupled dynamical systems with hyperbolic fast motions. *Discrete Contin. Dyn. Syst.* 13:1187-1201.
- [18] Leung, A. W. 2003. Asymptotically stable invariant manifold for coupled nonlinear parabolic-hyperbolic partial differential equations. *J. Differential Equations* 187:184-200.



- [19] Leung, A. W. 2005. Stable invariant manifolds for coupled Navier-Stokes and second-order wave systems. *Asymptotic Analysis* 43:339-357.
- [20] Liu, D. 2010. Strong convergence of principle of averaging for multiscale stochastic dynamical systems. *Commun. Math. Sci.* 8:999-1020.
- [21] Da Prato, G. and Zabczyk, J. 1992. *Stochastic Equations in Infinite Dimensions*. Cambridge:Cambridge University Press.
- [22] Rivera, J. E. M. and Racke, R. 1995. Smoothing properties, decay and global existence of solutions to nonlinear coupled systems of thermoelasticity type. *SIAM J. Math. Anal.* 26:1547-1563.
- [23] Veretennikov, A. Y. 1991. On the Averaging principle for systems of stochastic differential equations. *Mathematics of the USSR-Sbornik* 69:271-284.
- [24] Veretennikov, A. Y. 1999. On large deviations in the averaging principle for SDEs with full dependence. *Ann. Probab.* 27:284-296.
- [25] Wu, S. H., Chen, H. and Li, W.X. 2008. The local and global existence of the solutions of hyperbolic-parabolic system modeling biological phenomena. *Acta Mathematica Scientia* 28B: 101-116.
- [26] Xu, Y., Duan, J. and W. Xu. 2011. An averaging principle for stochastic dynamical systems with Lévy noise. *Physica D* 240:1395-1401.
- [27] Xu, Y., Guo, R., Liu, D., Zhang, H and Duan, J. 2014. Stochastic averaging principle for dynamical systems with fractional Brownian motion. *Discrete Contin. Dyn. Syst. Ser. B* 19:1197-1212.
- [28] Zhang, X. and Zuazua, E. 2007. Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction. *Arch. Rational Mech. Anal.* 184:49-120.